Interacting Fermions in Highly Elongated Harmonic Traps

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Quasi-one-dimensional two-component Fermi gases with effectively attractive and repulsive interactions are characterized for arbitrary interaction strength. The ground-state properties of the gas confined in highly elongated harmonic traps are determined within the local density approximation. For strong attractive effective interactions the existence of a molecular Tonks-Girardeau gas is predicted. The frequency of the lowest breathing mode is calculated as a function of the coupling strength for both attractive and repulsive interactions.

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The study of quasi-one-dimensional (Q1D) atomic quantum gases presents a very active area of research. So far, most of the experimental [1,2] and theoretical [3–7] investigations have been devoted to Q1D Bose gases and, in particular, to the strongly interacting Tonks-Girardeau gas, which can be mapped to a gas of non-interacting fermions [3,8,9]. Q1D two-component atomic Fermi gases have not been realized experimentally yet; however, their realization in highly elongated, needle-shaped traps is within reach of present-day techniques. The behavior of Q1D two-component Fermi gases can, if the confinement is chosen properly, be characterized to a very good approximation by an effective 1D coupling constant, g_{1D}, which encapsulates the interspecies atom-atom interaction strength. This coupling constant can be tuned to essentially any value, including zero and $\pm\infty$, by varying the 3D s-wave scattering length a_{3D} through application of an external magnetic field in the proximity of a Feshbach resonance.

The role of interactions in Q1D atomic Fermi gases has been studied mainly in connection with Luttinger liquid theory [10,11]. Recati *et al.* [11] investigate the properties of a two-component Fermi gas with *repulsive* interspecies interactions confined in highly elongated harmonic traps. In the limit of weak and strong coupling these authors relate the parameters of the Luttinger Hamiltonian, which describe the low-energy properties of the gas, to the microscopic parameters of the system. The prospect of realizing Luttinger liquids with cold fermionic atoms is fascinating since it allows detailed investigations of strongly correlated many-body systems, which play a central role in condensed matter physics [12], to be conducted.

In homogeneous 1D Fermi gases with attractive interactions, sound waves propagate with a well defined velocity, while spin waves exhibit a gap [13]. Furthermore, in the strong-coupling regime, the ground state is composed of bosonic molecules (consisting of two fermions with different spin), whose spatial size is much smaller than the average intermolecular distance [13]. Consequently, BCS-type equations have been discussed for effectively attractive 1D interactions [14]. The Q1D molecular Bose gas discussed here (see also Ref. [15]) has similarities with the formation of a molecular Bose-Einstein condensate (BEC) from a 3D Fermi sea close to a magnetic atom-atom Feshbach resonance [16].

This Letter investigates the properties of inhomogeneous Q1D two-component Fermi gases under harmonic confinement with *attractive and repulsive* interspecies interactions. Our study is based on the exact equation of state of a homogeneous 1D system of fermions with zero-range attractive [13,17] and repulsive [18] interactions treated within the local density approximation (LDA). We calculate the energy per particle, the size of the cloud, and the frequency of the lowest compressional mode as a function of the effective 1D coupling constant, including infinitely strong attractive and repulsive interactions. Our predictions for the size of the cloud and for the breathing mode frequency have immediate implications for experimental studies. It has been shown recently for Q1D Bose gases [2] that precise measurements of collective mode frequencies can provide evidence for beyond mean-field effects. For attractive interactions we discuss the cross-over from the weak- to the strong-coupling regime and point out the possibility of forming a mechanically stable molecular Tonks-Girardeau gas.

Consider a two-component atomic Fermi gas confined in a highly elongated trap. The fermionic atoms are assumed to belong to the same atomic species, that is, to have the same mass m, but to be trapped in different hyperfine states σ, where σ represents a generalized spin or angular momentum, $\sigma = \uparrow$ or $\downarrow$. The trapping potential is assumed to be harmonic and axially symmetric,

$$V_{\text{trap}} = \sum_{i=1}^{N} \frac{1}{2} m(\omega_\rho^2 \rho_i^2 + \omega_z^2 z_i^2). \tag{1}$$

Here, $\rho_i = \sqrt{x_i^2 + y_i^2}$ and z_i denote, respectively, the

radial and longitudinal coordinates of the ith atom; ω_ρ and ω_z denote, respectively, the angular frequencies in the radial and longitudinal directions; and N denotes the total number of atoms. We require the anisotropy parameter λ, $\lambda = \omega_z/\omega_\rho$, to be so small that the transverse motion is "frozen" to zero point oscillations. At zero temperature this implies that the Fermi energy associated with the longitudinal motion of the atoms in the absence of interactions, $\epsilon_F = N\hbar\omega_z/2$, is much smaller than the separation between the levels in the transverse direction, $\epsilon_F \ll \hbar\omega_\rho$. This condition is fulfilled if $\lambda \ll 1/N$. The outlined scenario can be realized experimentally with present-day technology using optical traps.

If the Fermi gas is kinematically in 1D, it can be described by an effective 1D Hamiltonian with contact interactions,

$$H = N\hbar\omega_\rho + H^0_{1D} + \sum_{i=1}^{N} \frac{1}{2} m\omega_z^2 z_i^2, \tag{2}$$

where

$$H^0_{1D} = -\frac{\hbar^2}{2m} \sum_{i=1}^{N} \frac{\partial^2}{\partial z_i^2} + g_{1D} \sum_{i=1}^{N_\uparrow} \sum_{j=1}^{N_\downarrow} \delta(z_i - z_j) \tag{3}$$

and $N = N_\uparrow + N_\downarrow$. This effective Hamiltonian accounts for the interspecies atom-atom interactions, which are parametrized by the 3D s-wave scattering length a_{3D}, through the effective 1D coupling constant g_{1D} [3],

$$g_{1D} = \frac{2\hbar^2 a_{3D}}{m a_\rho^2} \frac{1}{1 - A a_{3D}/a_\rho}, \tag{4}$$

but neglects the typically much weaker p-wave interactions. In Eq. (4), $a_\rho = \sqrt{\hbar/m\omega_\rho}$ is the characteristic oscillator length in the transverse direction and $A = |\zeta(1/2)|/\sqrt{2} \simeq 1.0326$. Alternatively, g_{1D} can be expressed through the effective 1D scattering length a_{1D}, $g_{1D} = -2\hbar^2/(m a_{1D})$, where

$$a_{1D} = -a_\rho \left(\frac{a_\rho}{a_{3D}} - A \right). \tag{5}$$

Figure 1 shows g_{1D} and a_{1D} as a function of the 3D s-wave scattering length a_{3D}, which can be varied continuously by the application of an external field. The effective 1D interaction is repulsive, $g_{1D} > 0$, for $0 < a_{3D} < a^c_{3D}$ ($a^c_{3D} = 0.9684 a_\rho$), and attractive, $g_{1D} < 0$, for $a_{3D} > a^c_{3D}$ and for $a_{3D} < 0$. By varying a_{3D}, it is possible to go adiabatically from the weakly interacting regime ($g_{1D} \sim 0$) to the strongly interacting repulsive regime ($g_{1D} \to +\infty$ or $a_{3D} \lesssim a^c_{3D}$), as well as from the weakly interacting regime to the strongly interacting attractive regime ($g_{1D} \to -\infty$ or $a_{3D} \gtrsim a^c_{3D}$) [19].

For two fermions with different spins the Hamiltonian H^0_{1D}, Eq. (3), supports one bound state with binding energy $\epsilon_{bound} = -\hbar^2/(m a^2_{1D})$ and spatial extent $\sim a_{1D}$ for $g_{1D} < 0$, and no bound state for $g_{1D} > 0$; that is, the molecular state becomes exceedingly weakly bound and spatially delocalized as $g_{1D} \to 0^-$ [3]. In the following we investigate the properties of a gas with N fermions, $N_\uparrow = N_\downarrow$, for both effectively *attractive and repulsive* 1D interactions *with and without* longitudinal confinement.

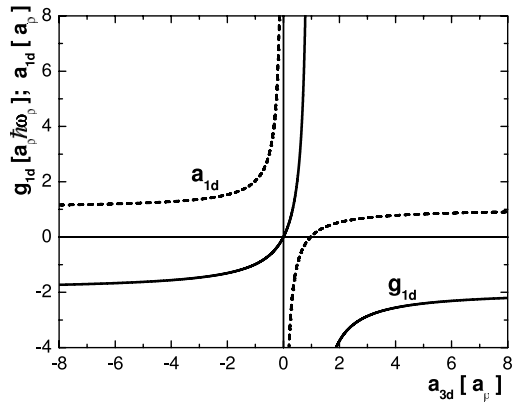

FIG. 1. Effective 1D coupling constant g_{1D} [solid line, Eq. (4)], together with effective 1D scattering length a_{1D} [dashed line, Eq. (5)] as a function of a_{3D}.

Consider the Hamiltonian H^0_{1D}, Eq. (3), which describes a homogeneous 1D two-component Fermi gas. The ground state energy E_{hom} of H^0_{1D} has been calculated exactly using Bethe's ansatz for attractive [17] and repulsive [18] interactions, and can be expressed in terms of the linear number density $n_{1D} = N/L$, where L is the size of the system,

$$\frac{E_{hom}}{N} = \frac{\hbar^2 n^2_{1D}}{2m} e(\gamma). \tag{6}$$

The dimensionless parameter γ is proportional to the coupling constant g_{1D}, $\gamma = m g_{1D}/(\hbar^2 n_{1D})$, while its absolute value is inversely proportional to the 1D gas parameter $n_{1D}|a_{1D}|$, $|\gamma| = 2/n_{1D}|a_{1D}|$. The function $e(\gamma)$ is obtained by solving a set of integral equations [20], which is similar to that derived by Lieb and Liniger [21] for 1D bosons with repulsive contact interactions. To obtain the energy per particle, Eq. (6), we solve these integral equations for $\gamma < 0$ [17] and for $\gamma > 0$ [18].

Figure 2 shows the energy per particle, E_{hom}/N (solid line), the chemical potential μ_{hom}, $\mu_{hom} = dE_{hom}/dN$ (dashed line), and the velocity of sound c (inset), which is obtained from the inverse compressibility $mc^2 = n_{1D}\partial\mu_{hom}/\partial n_{1D}$, as a function of the interaction strength γ. In the weak coupling limit, $|\gamma| \ll 1$, μ_{hom} is given by

$$\mu_{hom} = \frac{\pi^2}{4} \frac{\hbar^2 n^2_{1D}}{2m} + \gamma \frac{\hbar^2 n^2_{1D}}{2m} + \cdots, \tag{7}$$

where the first term on the right hand side is the energy of an ideal two-component atomic Fermi gas, and the second term is the mean-field energy, which accounts for interactions. The chemical potential increases with increasing γ, and reaches an asymptotic value for $\gamma \to \infty$

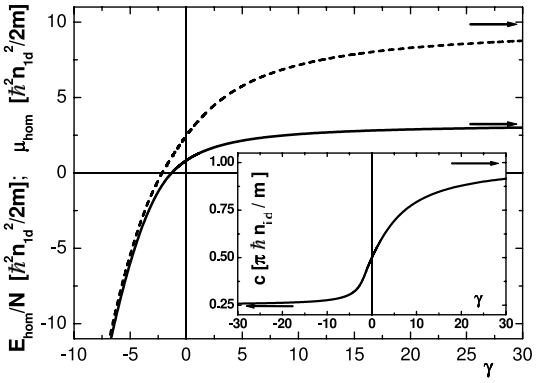

FIG. 2. $E_{\rm hom}/N$ (solid line), $\mu_{\rm hom}$ (dashed line), and c (inset) for a homogeneous two-component 1D Fermi gas as a function of γ (horizontal arrows indicate the asymptotic values of $E_{\rm hom}/N$, $\mu_{\rm hom}$, and c, respectively).

(indicated by a horizontal arrow in Fig. 2),

$$\mu_{\rm hom} = \pi^2 \frac{\hbar^2 n_{\rm 1D}^2}{2m} - \frac{16\pi^2 \ln(2)}{3\gamma} \frac{\hbar^2 n_{\rm 1D}^2}{2m} + \cdots. \quad (8)$$

The first term on the right hand side coincides with the chemical potential of a one-component ideal 1D Fermi gas with N atoms, the second term has been calculated in [11]. Interestingly, for $\gamma \gg 1$, the strong atom-atom repulsion between atoms with different spin plays the role of an effective Pauli principle [11].

For attractive interactions and large enough $|\gamma|$ the energy per particle is negative (see Fig. 2), reflecting the existence of a molecular Bose gas, which consists of $N/2$ diatomic molecules with binding energy $\epsilon_{\rm bound}$. Each molecule is composed of two atoms with different spin. In the limit $\gamma \to -\infty$, the chemical potential becomes

$$\mu_{\rm hom} = -\frac{\hbar^2}{2m a_{\rm 1D}^2} + \frac{\pi^2}{16} \frac{\hbar^2 n_{\rm 1D}^2}{2m} - \frac{\pi^2}{12\gamma} \frac{\hbar^2 n_{\rm 1D}^2}{2m} + \cdots. \quad (9)$$

The first term is simply $\epsilon_{\rm bound}/2$, one-half of the binding energy of the 1D molecule, while the second term is equal to half of the chemical potential of a bosonic Tonks-Girardeau gas with density $n_{\rm 1D}/2$, consisting of $N/2$ molecules with mass $2m$ [22]. Importantly, the compressibility remains positive for $\gamma \to -\infty$ [a horizontal arrow in the inset of Fig. 2 indicates the asymptotic value of c, $c = \pi\hbar n_{\rm 1D}/(4m)$], which implies that two-component 1D Fermi gases are mechanically stable even in the strongly attractive regime. In contrast, the ground state of 1D Bose gases with $g_{\rm 1D} < 0$ has negative compressibility [23] and is hence mechanically unstable.

Using the solutions for the homogeneous two-component 1D Fermi gas, we now describe the inhomogeneous gas, Eq. (2), within the LDA [5,6,11]. This approximation is applicable if the size R of the cloud is much larger than the harmonic oscillator length a_z in the longitudinal direction, $a_z = \sqrt{\hbar/m\omega_z}$, implying $\epsilon_F \gg \hbar\omega_z$ and $N \gg 1$. The chemical potential μ of the inhomogeneous system can be determined from the local equilibrium condition,

$$\mu = \mu_{\rm hom}[n_{\rm 1D}(z)] + \tfrac{1}{2} m\omega_z^2 z^2, \quad (10)$$

and the normalization condition $N = \int_{-R}^{R} n_{\rm 1D}(z)dz$, where z is measured from the center of the trap, $R = \sqrt{2\mu'/(m\omega_z^2)}$, and $\mu' = \mu$ for $g_{\rm 1D} > 0$ and $\mu' = \mu + |\epsilon_{\rm bound}|/2$ for $g_{\rm 1D} < 0$. The normalization condition can be reexpressed in terms of the dimensionless chemical potential $\tilde{\mu}$ and the dimensionless density $\tilde{n}_{\rm 1D}$ [$\tilde{\mu} = \mu'/(\hbar^2/2ma_{\rm 1D}^2)$ and $\tilde{n}_{\rm 1D} = |a_{\rm 1D}| n_{\rm 1D}$],

$$N \frac{a_{\rm 1D}^2}{a_z^2} = \int_0^{\tilde{\mu}} \frac{\tilde{n}_{\rm 1D}(\tilde{\mu} - x)}{\sqrt{x}} dx. \quad (11)$$

This expression emphasizes that the coupling strength is determined by $N a_{\rm 1D}^2/a_z^2$; $N a_{\rm 1D}^2/a_z^2 \gg 1$ corresponds to the weak coupling and $N a_{\rm 1D}^2/a_z^2 \ll 1$ to the strong coupling regime, irrespective of whether the interactions are attractive or repulsive [6].

Figure 3 shows the energy per particle and the size of the cloud as a function of the coupling strength $N a_{\rm 1D}^2/a_z^2$ for positive and negative $g_{\rm 1D}$ calculated within the LDA for an inhomogeneous two-component 1D Fermi gas. Compared to the noninteracting gas, for which $R = \sqrt{N} a_z$, R increases for repulsive interactions and decreases for attractive interactions. For $N a_{\rm 1D}^2/a_z^2 \ll 1$, R reaches the asymptotic value $\sqrt{2N} a_z$ for the strongly repulsive regime, $g_{\rm 1D} \to +\infty$, and the value $\sqrt{N/2} a_z$ for the strongly attractive regime, $g_{\rm 1D} \to -\infty$. The shrinking of the cloud for attractive interactions reflects the formation of tightly bound molecules. In the limit $g_{\rm 1D} \to -\infty$, the energy per particle approaches $\epsilon_{\rm bound}/2 + N\hbar\omega_z/8 + \hbar\omega_\rho$, indicating the formation of a molecular bosonic Tonks-Girardeau gas, consisting of $N/2$ molecules.

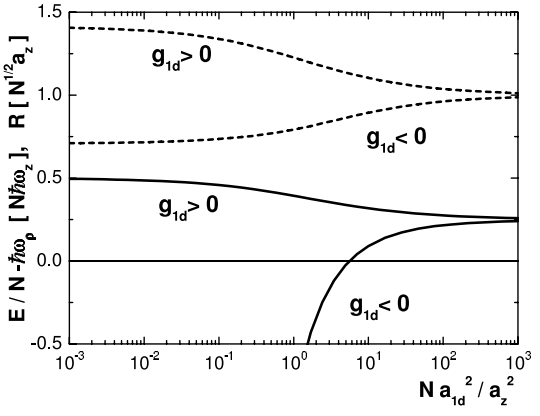

FIG. 3. Energy per particle, $E/N - \hbar\omega_\rho$ (solid lines), and size of the cloud, R (dashed lines), for an inhomogeneous two component 1D Fermi gas as a function of $N a_{\rm 1D}^2/a_z^2$ for repulsive ($g_{\rm 1D} > 0$) and attractive ($g_{\rm 1D} < 0$) interactions.

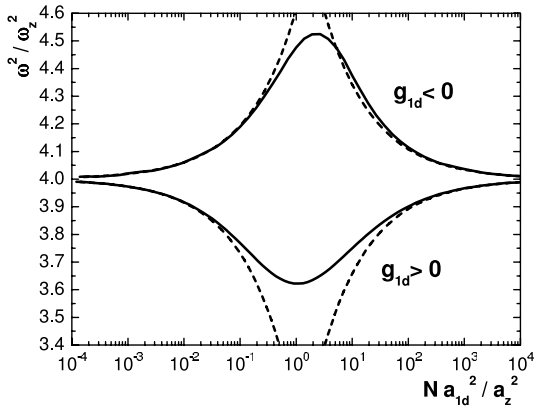

FIG. 4. Square of the lowest breathing mode frequency, ω^2, as a function of the coupling strength Na_{1D}^2/a_z^2 for an inhomogeneous two-component 1D Fermi gas with repulsive ($g_{1D} > 0$) and attractive ($g_{1D} < 0$) interactions determined numerically from Eq. (12) (solid lines). Dashed lines show analytic expansions (see text).

Using a sum rule approach, the frequency ω of the lowest compressional (breathing) mode of harmonically trapped 1D gases can be calculated from the mean-square size of the cloud $\langle z^2 \rangle$ [6],

$$\omega^2 = -2\frac{\langle z^2 \rangle}{d\langle z^2 \rangle / d\omega_z^2}. \tag{12}$$

In the weak and the strong coupling regimes ($Na_{1D}^2/a_z^2 \gg 1$ and $\ll 1$, respectively), $\langle z^2 \rangle$ has the same dependence on ω_z as the ideal 1D Fermi gas. Consequently, ω is in these limits given by $2\omega_z$. Solid lines in Fig. 3 show ω^2, determined numerically from Eq. (12), as a function of the interaction strength Na_{1D}^2/a_z^2. To gain further insight, we calculate the first correction $\delta\omega$ to the breathing mode frequency ω [$\omega = 2\omega_z(1 + \delta\omega/\omega_z + \cdots)$] analytically for weak repulsive and attractive interactions, as well as for strong repulsive and attractive interactions. For the weak coupling regime, we find $\delta\omega/\omega_z = \pm(4/3\pi^2)/(Na_{1D}^2/a_z^2)^{1/2}$, where the minus sign applies to repulsive interactions and the plus sign to attractive interactions. For the strong coupling regime, we find $\delta\omega/\omega_z = -[16\sqrt{2}\ln(2)/15\pi^2](Na_{1D}^2/a_z^2)^{1/2}$ for repulsive interactions and $\delta\omega/\omega_z = (8\sqrt{2}/15\pi^2)(Na_{1D}^2/a_z^2)^{1/2}$ for attractive interactions. Dashed lines in Fig. 4 show the resulting analytic expansions for ω^2, which describe the lowest breathing mode frequency quite well over a fairly large range of interaction strengths but break down for $Na_{1D}^2/a_z^2 \sim 1$.

In conclusion, we have investigated the crossover from weak to strong coupling of Q1D harmonically trapped two-component Fermi gases with both repulsive and attractive effective interactions. The frequency of the lowest breathing mode, which can provide an experimental signature of the crossover, is calculated. We predict the existence of a stable molecular Tonks-Girardeau gas in the strongly attractive regime.

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