Classical Phase-Space Descriptions of Continuous-Variable Teleportation

Carlton M. Caves[1,*] and Krzysztof Wódkiewicz[1,2,†]

[1]*Department of Physics and Astronomy, University of New Mexico, Albuquerque, New Mexico 87131-1156, USA*

[2]*Instytut Fizyki Teoretycznej, Uniwersytet Warszawski, Warszawa 00-681, Poland*

(Received 23 January 2004; published 22 July 2004)

The non-negative Wigner function of all quantum states involved in teleportation of Gaussian states using the standard continuous-variable teleportation protocol means that there is a local realistic phase-space description of the process. This includes the coherent states teleported up to now in experiments. We extend the phase-space description to teleportation of non-Gaussian states using the standard protocol and conclude that teleportation of non-Gaussian pure states with a fidelity of 2/3 is a "gold standard" for this kind of teleportation.

DOI: 10.1103/PhysRevLett.93.040506 PACS numbers: 03.67.Hk, 03.65.Ta, 42.50.Dv

Quantum teleportation is a process that can transfer an arbitrary quantum state from a system held by one party, usually called Alice, to a system held by a second party, usually called Bob. The process requires a pair of systems, shared by Alice and Bob, in an entangled state—the entangled resource—and a "small" amount of classical information transmitted from Alice to Bob. Originally proposed for qubit states [1], teleportation protocols were later extended to states of a system described by continuous phase-space variables, such as a massive particle or a mode of an optical field [2,3]. This continuous-variable teleportation protocol was implemented in an experiment that teleported a coherent state of an optical-frequency electromagnetic mode with fidelity 0.58 ± 0.02 [4]. Two recent experiments have improved the experimental fidelity of the teleported coherent state to values of 0.64 ± 0.02 [5] and 0.61 ± 0.02 [6].

In the standard continuous-variable teleportation protocol [3], Alice and Bob share an entangled Gaussian state of two modes, A and B, which have annihilation operators a and b; this entangled resource is ideally a two-mode squeezed state [7]. The state to be teleported is the pure [8] state $\rho = |\psi\rangle\langle\psi|$ of a mode V in Alice's possession, which has annihilation operator v. The protocol consists of (i) Alice's measuring two (commuting) joint quadrature components of modes V and A, specifically the Hermitian real and imaginary parts of the operator $v + a^\dagger$, (ii) Alice's communicating the (complex) result ξ to Bob, and (iii) Bob's displacing mode B by ξ. The efficacy of the protocol is quantified by the fidelity between the output state of mode B and the input state $|\psi\rangle$, averaged over the possible measurement results.

Experiments to date have teleported only coherent states. It is generally believed, though not proved, that teleporting coherent states with average fidelity $\mathcal{F} > 1/2$ requires an entangled resource [9]. It has thus been argued that teleportation of coherent states with fidelities above $1/2$ constitutes truly quantum teleportation [9,10]. Using a variety of arguments, other workers have contended that $\mathcal{F} = 2/3$ is the appropriate boundary between classical and quantum teleportation [11,12]. While acknowledging the need for an entangled resource for teleporting coherent states with fidelity $\mathcal{F} > 1/2$ (indeed, we provide additional evidence), we add a fresh perspective by investigating whether the entangled resource is used in a way that can be accounted for by a classical phase-space description. When such a description exists, it provides a local realistic hidden-variable model for the teleportation protocol.

Our investigation is motivated by the fact that the coherent states and the Gaussian entangled resource used in the experiments have non-negative Wigner functions [13], which are phase-space probability distributions that provide a classical description of measurements of the quadrature components. A non-negative Wigner function does not give a classical description of measurements other than those of quadrature components; specific such measurements on an entangled Gaussian state cannot be given a local realistic description and thus violate Bell inequalities [14]. Since the standard protocol uses only quadrature measurements, however, we conclude that for teleporting coherent states—or any Gaussian state—using the standard teleportation protocol, the non-negative Wigner function of the three modes gives a classical, local realistic description for all fidelities. This means that all the experiments to date—and any such experiment that teleports coherent states, no matter what fidelity is achieved—can be accounted for in terms of purely classical correlations, with no need for a quantum-mechanical explanation.

To find situations where the Wigner function does not provide a classical phase-space description of the standard protocol, we must look to teleportation of non-Gaussian states, which (for pure states) have Wigner functions that take on negative values [15]. To accommodate non-Gaussian states, we extend our hidden-variable model by allowing (i) Alice to substitute a randomly displaced state with a non-negative Wigner function in place of the non-Gaussian state and (ii) Alice and Bob to cheat by teleporting this new "smeared-out" state with

 0031-9007/04/93(4)/040506(4)$22.50

perfect fidelity. Teleportation of non-Gaussian pure states with fidelity $\mathcal{F} \geq 2/3$ cannot be accommodated within this extended hidden-variable model, thus making a fidelity of 2/3 a "gold standard" for teleportation of non-Gaussian pure states.

We begin with a brief Wigner-function-based review of the teleportation protocol. The state ρ_{AB} of modes A and B has Wigner function $W_{AB}(\alpha, \beta)$, which is a quasidistribution for the c-number complex amplitudes α and β corresponding to the annihilation operators a and b [16]. In the standard protocol, $W_{AB}(\alpha, \beta)$ is a Gaussian, but for the present, we allow it to be a general Wigner function. The (pure [8]) state $\rho = |\psi\rangle\langle\psi|$ of mode V has Wigner function $W_\rho(\nu)$, where ν is the c-number complex amplitude corresponding to annihilation operator v. The overall Wigner function of the three modes is $W_\rho(\nu)W_{AB}(\alpha, \beta)$. The state of mode B after a measurement of $v + a^\dagger$ that yields result ξ has Wigner function

$$W'(\beta \mid \xi) = \frac{1}{p(\xi)} \int d^2\nu d^2\alpha \times \delta(\nu + \alpha^* - \xi) W_\rho(\nu) W_{AB}(\alpha, \beta), \quad (1)$$

where

$$p(\xi) = \int d^2\nu d^2\alpha d^2\beta \delta(\nu + \alpha^* - \xi) W_\rho(\nu) W_{AB}(\alpha, \beta) \quad (2)$$

is the probability to obtain result ξ.

Having received result ξ from Alice, Bob displaces the complex amplitude of mode B by ξ, yielding a state $\rho_{\text{out}}(\xi)$ with Wigner function $W_{\text{out}}(\beta \mid \xi) = W'(\beta - \xi \mid \xi)$. The fidelity of this output state and the input state is $F(\xi) = \langle\psi|\rho_{\text{out}}(\xi)|\psi\rangle$. We are interested in the average of this fidelity over all measurement results, $\mathcal{F} = \int d^2\xi p(\xi) F(\xi) = \langle\psi|\overline{\rho}_{\text{out}}|\psi\rangle$, where $\overline{\rho}_{\text{out}} = \int d^2\xi p(\xi) \rho_{\text{out}}(\xi)$ is the average output state, having Wigner function

$$W_{\overline{\rho}_{\text{out}}}(\beta) = \int d^2\xi p(\xi) W_{\text{out}}(\beta \mid \xi) = \int d^2\nu G(\nu) W_\rho(\beta - \nu). \quad (3)$$

Here

$$G(\nu) = \int d^2\alpha d^2\beta \delta(\beta + \alpha^* - \nu) W_{AB}(\alpha, \beta) \quad (4)$$

is the (non-negative) probability to obtain result ν in a measurement of $b + a^\dagger$ on modes A and B. Equation (3) shows that the average output state is a mixture of displaced input states,

$$\overline{\rho}_{\text{out}} = \int d^2\nu G(\nu) D(\nu) \rho D^\dagger(\nu), \quad (5)$$

where $D(\nu)$ is the displacement operator.

We can now write the average output fidelity in two complementary forms,

$$\mathcal{F} = \int d^2\nu G(\nu) |C_\rho(\nu)|^2 = \pi \int d^2\beta d^2\nu G(\beta - \nu) W_\rho(\beta) W_\rho(\nu), \quad (6)$$

where

$$C_\rho(\nu) = \langle\psi|D(\nu)|\psi\rangle = \int d^2\mu W_\rho(\mu) e^{\nu\mu^* - \nu^*\mu}, \quad (7)$$

the symmetrically ordered (Wigner-Weyl) characteristic function of the input state [17], is the Fourier transform of the Wigner function. The first form in Eq. (6) comes directly from Eq. (5), and the second from writing the fidelity as an overlap of the Wigner functions for the input and average output states. The effect of the initial state of modes A and B on the average fidelity is contained wholly in the marginal distribution $G(\nu)$. High-fidelity teleportation occurs when $G(\nu)$ is very narrow, i.e., when the quadrature components contained in $b + a^\dagger$ are sharp, expressing a particular kind of correlation between modes A and B. Using the Fourier transform (7), we can derive two other, equivalent forms for the average fidelity,

$$\mathcal{F} = \pi \int d^2\beta d^2\nu \tilde{G}(\beta - \nu) W_\rho(\beta) W_\rho(\nu) = \int d^2\nu \tilde{G}(\nu) |C_\rho(\nu)|^2, \quad (8)$$

where

$$\tilde{G}(\nu) = \int d^2\mu G(\mu) e^{\nu\mu^* - \nu^*\mu} \quad (9)$$

is the Fourier transform of $G(\mu)$.

Before proceeding to the standard protocol and our hidden-variable models, we pause here to demonstrate the one technical result we need. We wish to find the maximum value of the integral

$$I = \int d^2\alpha d^2\beta e^{-t|\alpha - \beta|^2/2} W_{AB}(\alpha, \beta), \qquad t \geq 0, \quad (10)$$

over the Wigner function $W_{AB}(\alpha, \beta)$ of a joint state ρ_{AB} of modes A and B. Introducing annihilation operators $c = (a + b)/\sqrt{2}$ and $d = (a - b)/\sqrt{2}$, with corresponding c-number variables γ and δ, we can rewrite I as

$$I = \int d^2\gamma d^2\delta e^{-t|\delta|^2} W_{CD}(\gamma, \delta) = \int d^2\delta e^{-t|\delta|^2} W_D(\delta), \quad (11)$$

where $W_{CD}(\gamma, \delta) = W_{AB}(\alpha, \beta)\sigma$ is the Wigner function written in terms of modes C and D, and $W_D(\delta)$ is the Wigner function for mode D alone. The integral can now be thought of as the expectation value, $\text{tr}(A_t \rho_D) = \text{tr}(A_t \rho_{AB})$, of the D-mode operator A_t whose symmetrically ordered associated function [17] is $e^{-t|\delta|^2}$, this operator being $A_t = (1 + t/2)^{-1}[(1 - t/2)/(1 + t/2)]^{d^\dagger d}$.

The integral I being the expectation value of A_t, I is bounded above by the largest eigenvalue of A_t. Since A_t is diagonal in the number-state basis, with eigenvalues that decrease in magnitude with the number of quanta, we have

$$I_{\max} = (\text{largest eigenvalue of } A_t) = \frac{1}{1 + t/2}, \qquad (12)$$

with the maximum achieved if and only if ρ_{AB} is the vacuum state for mode D.

For the case that ρ_{AB} is a pure product state, $|\Psi\rangle = |\psi_A\rangle \otimes |\psi_B\rangle$, which turns out to be the case of interest here, the condition for achieving the maximum becomes $d|\Psi\rangle = 0$ or, equivalently, $a|\psi_A\rangle \otimes |\psi_B\rangle = |\psi_A\rangle \otimes b|\psi_B\rangle$, from which it follows that $a|\psi_A\rangle = |\psi_A\rangle\langle\psi_B|b|\psi_B\rangle$ and $b|\psi_B\rangle = |\psi_B\rangle\langle\psi_A|a|\psi_A\rangle$, implying that $|\psi_A\rangle$ and $|\psi_B\rangle$ are identical coherent states. Thus the only pure product states that achieve the maximum in Eq. (12) are products of identical coherent states.

We can use Eq. (12) to get one interesting result immediately: The maximum average fidelity for teleporting a coherent state using the standard protocol, but with a separable state for modes A and B, is 1/2. To show this, suppose first that modes A and B are in a pure product state, with factorizable Wigner function $W_A(\alpha)W_B(\beta)$. The characteristic function for any coherent state satisfies $|C_{\rm coh}(\nu)|^2 = e^{-|\nu|^2}$, so we can use Eq. (4) and the first form in Eq. (6) to write the average fidelity as

$$\mathcal{F} = \int d^2\alpha d^2\beta e^{-|\alpha-\beta|^2} W_A(-\alpha^*)W_B(\beta). \qquad (13)$$

Here $W_A(-\alpha^*)$ is the Wigner function for the time-reversed, parity-inverted state of mode A. The $t = 2$ general bound (12) implies $\mathcal{F} \le 1/2$, with equality if and only if mode A is in a coherent state $|\alpha\rangle$ and mode B is in the time-reversed, parity-inverted coherent state $|-\alpha^*\rangle$.

Now suppose modes A and B are initially in a separable state, thus having a pure product-state ensemble decomposition. The fidelity is the average over the pure product-state ensemble, which shows that the fidelity is still bounded above by 1/2, with equality if and only if the separable state is a mixture of product states of the form $|\alpha\rangle \otimes |-\alpha^*\rangle$. This does not show that 1/2 is the maximum fidelity for coherent-state teleportation in the absence of entanglement, since the result applies only to the standard protocol, but it is an additional piece of evidence, distinct from the results reported in Ref. [10].

We now take up again our analysis of the standard teleportation protocol, assuming that modes A and B are in a Gaussian state with Wigner function

$$W_{AB}(\alpha, \beta) = \frac{4(\mathsf{c}^2 - \mathsf{s}^2)}{\pi^2} e^{-2\mathsf{c}(|\alpha|^2+|\beta|^2)-2\mathsf{s}(\alpha\beta+\alpha^*\beta^*)}, \quad (14)$$

where c and s satisfy $|\mathsf{s}| < \mathsf{c} \le \sqrt{1+\mathsf{s}^2}$. This state is pure if and only if $\mathsf{c} = \sqrt{1+\mathsf{s}^2}$, in which case the state becomes a two-mode squeezed state with $\mathsf{c} = \cosh 2r$ and $\mathsf{s} = \sinh 2r$, where r is the squeeze parameter [7]. The state (14) is separable if and only if $\mathsf{c} \le 1 - |\mathsf{s}|$ [18].

For the Wigner function (14), the distribution (4) is a Gaussian,

$$G(\nu) = \frac{\mathsf{c}+\mathsf{s}}{\pi} e^{-(\mathsf{c}+\mathsf{s})|\nu|^2} = \frac{2}{\pi t} e^{-2|\nu|^2/t}, \qquad (15)$$

where $t \equiv 2/(\mathsf{c}+\mathsf{s})$ is the single parameter needed to characterize the fidelity that can be achieved with this entanglement resource. The Wigner function (3) of the average output state is the $(s = -t)$-ordered quasidistribution, $W_\rho^{(s)}(\nu)$ [17], of the input state:

$$W_{\overline{\rho}_{\rm out}}(\beta) = \frac{2}{\pi t}\int d^2\nu e^{-2|\beta-\nu|^2/t} W_\rho(\nu) = W_\rho^{(s)}(\nu). \quad (16)$$

For $t = 0$, $G(\nu)$ is a δ function, and the output state is identical to the input state (perfect teleportation). For $t = 1$, the Wigner function of the average output state is the Husimi Q distribution of the input state, i.e., $W_{\overline{\rho}_{\rm out}}(\beta) = W_\rho^{(-1)}(\beta) = Q_\rho(\beta) = \langle\beta|\rho|\beta\rangle/\pi$.

For $0 \le t < 2$ ($\mathsf{c} > 1 - \mathsf{s}$), the state (14) is entangled, with the right sort of correlations for this protocol, these correlations decreasing as t increases. At $t = 2$, the state passes through the separability boundary $\mathsf{c}+\mathsf{s} = 1$, and for $t \ge 2$, the state either is separable ($\mathsf{c} \le 1 - |\mathsf{s}|$) or, though entangled ($\mathsf{c} > 1 + \mathsf{s}$), has the wrong sort of correlations for this protocol.

The average fidelity of Eqs. (6) and (8) now becomes

$$\begin{aligned}\mathcal{F}_\rho(t) &= \frac{2}{\pi t}\int d^2\nu e^{-2|\nu|^2/t}|C_\rho(\nu)|^2 \\ &= \frac{2}{t}\int d^2\beta d^2\nu e^{-2|\beta-\nu|^2/t} W_\rho(\beta)W_\rho(\nu) \\ &= \int d^2\beta d^2\nu e^{-t|\beta-\nu|^2/2} W_\rho(\beta)W_\rho(\nu) \\ &= \frac{1}{\pi}\int d^2\nu e^{-t|\nu|^2/2}|C_\rho(\nu)|^2. \end{aligned} \qquad (17)$$

The second form is the overlap of the Wigner function and the s-ordered quasidistribution for the input state. The first two derivatives of the last form show that $\mathcal{F}_\rho(t)$ is a strictly decreasing, strictly concave function of t. These forms also show that the average fidelity obeys the scaling relation $\mathcal{F}_\rho(t) = 2\mathcal{F}_\rho(4/t)/t$, which again draws attention to the separability boundary at $t = 2$.

Given that $|C_{\rm coh}(\nu)|^2 = e^{-|\nu|^2}$, the average fidelity for teleporting a coherent state is $\mathcal{F}_{\rm coh}(t) = (1 + t/2)^{-1}$. For number states $|n\rangle$, whose Wigner functions take on negative values (except for $n = 0$), we have obtained an analytic formula for a generating function

$$\begin{aligned}\mathcal{F}(\lambda, t) &= \sum_{n=0}^{\infty} \lambda^n \mathcal{F}_{|n\rangle\langle n|}(t) \\ &= \frac{1}{\sqrt{(1+t/2)^2 - 2\lambda(1+t^2/4) + \lambda^2(1-t/2)^2}}. \end{aligned} \qquad (18)$$

The resulting fidelity for teleporting a number state is

$$\mathcal{F}_{|n\rangle\langle n|}(t) = \frac{(1-t/2)^n}{(1+t/2)^{n+1}} P_n\left(\frac{1+t^2/4}{1-t^2/4}\right), \qquad (19)$$

where $P_n(x)$ is a Legendre polynomial. This gives $\mathcal{F}_{|n\rangle\langle n|}(1) = 2P_n(5/3)/3^{n+1}$ and $\mathcal{F}_{|n\rangle\langle n|}(2) = (2n)!/2^{2n+1}(n!)^2$.

We now return to our Wigner-function-based discussion of hidden-variable models for teleportation. For Gaussian input states and for the two-mode entangled resource (14), all the Wigner functions are non-negative [13], so they provide a classical phase-space description—and hence a local hidden-variable description—for this kind of teleportation, no matter what fidelity is achieved. The hidden variables are the quadrature components of all the modes, and the overall Wigner function is a probability distribution for these hidden variables.

All non-Gaussian input pure states have Wigner functions that take on negative values (Hudson-Piquet theorem [15]) and thus cannot be incorporated in the simple hidden-variable model. To see what can be achieved within a classical phase-space description, suppose that before performing the teleportation protocol, Alice "kicks" the input state ρ randomly in phase space. The random kick is described by a Gaussian so that the average state after the kick is

$$\rho' = \frac{2}{\pi t}\int d^2\nu e^{-2|\nu|^2/t} D(\nu)\rho D^\dagger(\nu). \qquad (20)$$

We choose the kicking strength t to be the minimum value necessary to make ρ' have a non-negative Wigner function, thus giving ρ' a classical phase-space description and allowing it to be incorporated within our hidden-variable model. For all non-Gaussian states, this minimum kicking strength is one vacuum unit, i.e., $t = 1$ [19], implying that ρ' is the state whose Wigner function, $W_{\rho'}(\nu) = W_\rho^{(-1)}(\nu) = Q_\rho(\nu)$, is the Q function of the original state ρ. Further suppose that Alice and Bob cheat by teleporting ρ' with perfect fidelity. Then the fidelity of the overall process is the overlap of the Wigner and Q functions of ρ, i.e., the $t = 1$ fidelity (17) of the standard protocol. Notice that bigger kicks ($t > 1$) would give smaller fidelity, making clear why we choose the smallest kicking strength consistent with giving ρ' a non-negative Wigner function.

These considerations, coupled with wanting to know the maximum teleportation fidelity for a given entangled resource t, motivate us to find the maximum value of the average fidelity $\mathcal{F}_\rho(t)$ over all input pure states ρ, $t = 1$ being the value relevant for our hidden-variable model. The task can be restated as finding the pure state ρ that maximizes the overlap of the Wigner function and the s-ordered quasidistribution $W_\rho^{(s)}(\nu)$. We apply the bound (12) to the third form of the average fidelity (17), in this case maximizing over pure product states $\rho \otimes \rho$. The resulting maximum is $\mathcal{F}_{\max}(t) = (1 + t/2)^{-1}$, achieved if and only if ρ is a coherent state.

Each non-Gaussian input state ρ has its own threshold fidelity, $\mathcal{F}_\rho(1) < \mathcal{F}_{\max}(1) = 2/3$, below which its teleportation can be accommodated within our extended phase-space hidden-variable model and above which it cannot. Thus teleportation of a non-Gaussian state with fidelity exceeding $\mathcal{F}_\rho(1)$ is required to rule out an explanation in terms of classical phase-space correlations. A fidelity of $2/3$ emerges as a gold standard for continuous-variable teleportation in the sense that teleportation of any non-Gaussian pure state with $\mathcal{F} \geq 2/3$ cannot be fitted within our extended hidden-variable model. This conclusion applies only to our phase-space-based hidden-variable model; we have not shown that there is no local hidden-variable model that can accommodate teleportation fidelities of $2/3$ or above.

This work was supported in part by the U.S. Office of Naval Research Grant No. N00014-03-1-0426 (C. M. C.) and by KBN Grant No. 2PO3B02123 (K. W.).

*Electronic address: caves@info.phys.unm.edu
†Electronic address: wodkiew@fuw.edu.pl

[1] C. H. Bennett *et al.*, Phys. Rev. Lett. **70**, 1895 (1993).
[2] L. Vaidman, Phys. Rev. A **49**, 1473 (1994).
[3] S. L. Braunstein and H. J. Kimble, Phys. Rev. Lett. **80**, 869 (1998).
[4] A. Furusawa *et al.*, Science **282**, 706 (1998).
[5] W. P. Bowen *et al.*, Phys. Rev. A **67**, 032302 (2003).
[6] T. C. Zhang *et al.*, Phys. Rev. A **67**, 033802 (2003).
[7] B. L. Schumaker, Phys. Rep. **135**, 317 (1986).
[8] We restrict attention to teleporting pure states.
[9] S. L. Braunstein, C. A. Fuchs, and H. J. Kimble, J. Mod. Opt. **47**, 267 (2000).
[10] S. L. Braunstein, C. A. Fuchs, H. J. Kimble, and P. van Loock, Phys. Rev. A **64**, 022321 (2001).
[11] T. C. Ralph and P. K. Lam, Phys. Rev. Lett. **81**, 5668 (1998); T. C. Ralph, P. K. Lam, and R. E. S. Polkinghorne, J. Opt. B Quantum Semiclassical Opt. **1**, 483 (1999); R. Schnabel *et al.*, Opt. Sectrosc. **94**, 651 (2003).
[12] F. Grosshans and P. Grangier, Phys. Rev. A **64**, 010301 (2001).
[13] J. S. Bell, *Speakable and Unspeakable in Quantum Mechanics* (Cambridge University, Cambridge, England, 1987), Chap. 21; D. F. Walls and G. J. Milburn, *Quantum Optics* (Springer, Berlin, 1994).
[14] K. Banaszek and K. Wódkiewicz, Phys. Rev. A **58**, 4345 (1999); Phys. Rev. Lett. **82**, 2009 (1999); Z.-B. Chen, J.-W. Pan, G. Hou, and Y.-D. Zhang, Phys. Rev. Lett. **88**, 040406 (2002).
[15] See, for example, W. P. Schleich, *Quantum Optics in Phase Space* (Wiley-VCH, Weinheim, 2001), Chap. 3.
[16] For phase-space methods using the Wigner function, see, for example, C.W. Gardiner and P. Zoller, *Quantum Noise* (Springer, Berlin, 2000).
[17] K. E. Cahill and R. J. Glauber, Phys. Rev. **177**, 1857 (1969); **177**, 1882 (1969).
[18] See, for example, B.-G. Englert and K. Wódkiewicz, Int. J. Quantum Inf. **1**, 153 (2003), and references therein.
[19] N. Lütkenhaus and S. M. Barnett, Phys. Rev. A **51**, 3340 (1995).