Pattern Control via Multifrequency Parametric Forcing

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We use symmetry considerations to investigate control of a class of resonant three-wave interactions relevant to pattern formation in weakly damped, parametrically forced systems near onset. We classify and tabulate the most important damped, resonant modes and determine how the corresponding resonant triad interactions depend on the forcing parameters. The relative phase of the forcing terms may be used to enhance or suppress the nonlinear interactions. We compare our symmetry-based predictions with numerical and experimental results for Faraday waves. Our results suggest how to design multifrequency forcing functions that favor chosen patterns in the lab.

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Three-wave resonance plays a crucial role in the dynamics of many systems including fluids [1], plasmas [2] and optics [3], often providing the building blocks of complex nonlinear states. One can therefore influence a wide range of nonlinear states by controlling the underlying *resonant triad* interactions. In this Letter, we pursue the implications of this idea for pattern formation in spatially extended, parametrically forced systems.

Periodically modulated systems are abundant in nature. Likewise, parametric forcing has become a common tool for experimentalists. In fluid [4] and granular [5] systems this forcing typically takes the form of vertical vibrations, which, at sufficient strength, lead to patterns of standing waves on the free surface. In photosensitive reaction-diffusion systems the parametric forcing may take the form of periodic pulses of light [6], while recent experiments on ferrofluids have used a periodically modulated magnetic field [7]. A rich variety of states has been observed in these systems, including superlattice patterns, quasipatterns, and localized structures.

A prototypical model for parametric instability, the Mathieu equation captures several common features of parametrically forced systems [8]. With a forcing frequency ω, one expects a number of frequency-locked *resonance tongues* associated with natural modes of frequency $N\frac{\omega}{2}$, $N \in \mathbb{Z}^+$. In damped systems the $N = 1$ "subharmonic" mode typically sets in first and dominates the dynamics near onset. Nonetheless, secondary resonance tongues still play an important role in the selection of nonlinear states, particularly in weakly damped systems, and even more so when the forcing is not strictly sinusoidal (different Fourier components in the forcing will favor different modes).

We focus on controlling a class of resonant triad interactions involving two parametrically driven critical modes and a third, linearly damped mode whose wavelength determines a resonant angle $\theta_{\rm res}$; see Fig. 1. The damped mode may draw energy from the excited modes, creating an "antiselection" mechanism that suppresses such triads in favor of patterns that avoid the resonant angle [9], or it may serve as an energy source, and instead enhance associated patterns [10]. We describe a systematic method for ranking triad interactions according to their importance in the weakly damped limit, and, in many cases, for *controlling* their effect via the multifrequency forcing function

$$F(t) = f_m e^{im\omega t} + f_n e^{in\omega t} + f_p e^{ip\omega t} + \cdots + \text{c.c.}, \quad (1)$$

characterized by the choice of commensurate frequencies $(m\omega, n\omega, p\omega, \ldots)$, amplitudes $(|f_m|, |f_n|, |f_p|, \ldots)$, and phases $(\phi_m, \phi_n, \phi_p, \ldots)$, where $\phi_u = \arg(f_u)$.

Our main result is a table summarizing, for up to three forcing frequencies, which damped modes are likely to be important, the manner in which their coupling with critical modes depends on the forcing parameters and, in many cases, the overall qualitative effect (enhancing or suppressing) they have on associated patterns. This table

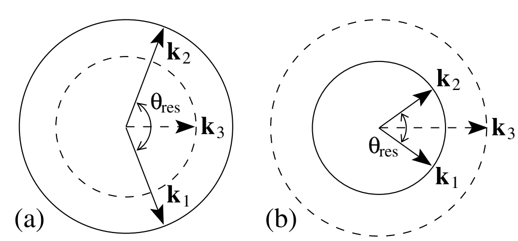

FIG. 1. Fourier space diagram of resonant triads composed of two critical modes ($|\mathbf{k}_1| = |\mathbf{k}_2| = k_1$) and a damped mode ($\mathbf{k}_3 = \mathbf{k}_1 + \mathbf{k}_2$, $|\mathbf{k}_3| = k_3$) with (a) $k_3 < k_1$, (b) $k_1 < k_3 < 2k_1$.

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indicates how patterns may be selected, or "designed," by judiciously adjusting the Fourier content of $F(t)$. The theory pertains to spatially extended systems with weak damping, but otherwise is quite general, relying only on the spatial and temporal symmetries of the problem, and on the dispersion relation. The details of the particular governing equations, which are unavailable in many cases, are not needed. We demonstrate the utility of our results in two examples relevant to recent experiments on pattern formation on vertically shaken fluid layers (Faraday waves).

Our analysis uses the (broken) symmetries of time translation, time reversal, and Hamiltonian structure, as in [11]. We take Ω to be the dominant frequency of the damped $\mathbf{k}_3$ mode of Fig. 1; the interesting values of Ω are determined below. Time is rescaled so that $\omega = 1$, and we denote by m the component of the forcing that drives the critical modes, which therefore have dominant frequency $m/2$. All other modes are linearly damped. We expand the physical variables (e.g., surface height h for Faraday waves) in terms of six traveling waves (TW):

$$h(\mathbf{x}, t) = \sum_{j=1}^{3} \sum_{\pm} Z_j^{\pm}(t)\, e^{i(\mathbf{k}_j \cdot \mathbf{x} \pm \varpi_j t)} + \text{c.c.} + \cdots, \qquad (2)$$

where $\varpi_1 = \varpi_2 = m/2$ and $\varpi_3 = \Omega$. The TW amplitude equations must respect the spatial symmetries of the problem (translation, reflection through $\mathbf{k}_3$, inversion through the origin) and the temporal symmetries

$$T_\tau: \; Z_j^{\pm} \to e^{\pm i\varpi_j \tau} Z_j^{\pm}, \; f_u \to e^{iu\tau} f_u, \qquad (3a)$$

$$\kappa: \; Z_j^{\mp} \leftrightarrow \bar{Z}_j^{\mp}, \; (t, \gamma) \to -(t, \gamma), \; f_u \to \bar{f}_u, \qquad (3b)$$

representing time translation and reversal, respectively. Here u denotes any of the frequencies $\{m, n, p, \cdots\}$ in (1) and γ is a dimensionless damping parameter.

The equivariant TW equations take the form

$$\dot{Z}_1^{+} = \upsilon Z_1^{+} - \lambda f_m Z_1^{-} + \mathcal{Q}_1(Z_2^{\pm}, Z_3^{\pm}) + \cdots, \qquad (4a)$$

$$\dot{Z}_3^{+} = \varrho Z_3^{+} - \mu F_{2\Omega} Z_3^{-} + \mathcal{Q}_3(Z_1^{\pm}, Z_2^{\pm}) + \cdots, \qquad (4b)$$

with the remaining four equations related by symmetry. The parametric forcing term $F_{2\Omega}$ represents products of the f_u whose frequencies sum to 2Ω ($F_{2\Omega} = f_{2\Omega}$ when 2Ω forcing is present). The resonant terms

$$\mathcal{Q}_1 = (Q_1 Z_3^{+} + Q_2 Z_3^{-})\bar{Z}_2^{+} + (Q_3 Z_3^{+} + Q_4 Z_3^{-})\bar{Z}_2^{-},$$

$$\mathcal{Q}_3 = Q_5 Z_1^{+} Z_2^{+} + Q_6 (Z_1^{+} Z_2^{-} + Z_1^{-} Z_2^{+}) + Q_7 Z_1^{-} Z_2^{-},$$

appear at quadratic order with coefficients Q_ℓ that, according to T_τ, must transform as $(Q_1, \bar{Q}_5) \sim e^{i(m-\Omega)\tau}$, $(Q_2, Q_7) \sim e^{i(m+\Omega)\tau}$, $(Q_3, \bar{Q}_4, \bar{Q}_6) \sim e^{-i\Omega\tau}$. The dependence of the linear coefficients υ, λ, etc., on γ and the f_u, is also determined by the symmetries T_τ and κ, e.g.,

$$\lambda = i\lambda_i + \lambda_r \gamma + i c_\gamma \gamma^2 + i \sum_u c_u |f_u|^2 + \mathcal{O}(\gamma^3, \gamma |f_u|^2),$$

where $\lambda_i, \lambda_r, c_\gamma, c_u \in \mathbb{R}$. With k_1 and k_3 defined by the local minima of the neutral stability curves the detunings $\mathrm{Im}(\upsilon)$ and $\mathrm{Im}(\varrho)$ must vanish at $\gamma = f_u = 0$. We need only the leading term in each coefficient:

$$\upsilon = -\upsilon_r \gamma, \qquad \varrho = -\varrho_r \gamma, \qquad \lambda = i\lambda_i,$$

$$\mu = i\mu_i, \qquad Q_\ell = i q_\ell F_\ell,$$

where $\upsilon_r, \varrho_r > 0$ (they correspond to damping terms), $\mu_i, q_\ell \in \mathbb{R}$ and F_ℓ denotes an appropriate product of the f_u (or unity). It follows that the critical value of $|f_m|$ is $\mathcal{O}(\gamma)$ and, since all $|f_u|$ are assumed to be related by a finite γ-independent ratio, $|f_u| \sim \gamma$.

For a given damped mode, we determine how the resonant coefficients Q_ℓ depend on the f_u, then apply a standard reduction procedure at the bifurcation to standing waves (SW). At cubic order we obtain

$$\dot{A}_1 = \lambda A_1 + A_1(a|A_1|^2 + (b_0 + b_{\text{res}})|A_2|^2), \qquad (5)$$

where $A_{1,2}$ are the slowly varying complex amplitudes of the SW near onset, and all coefficients are real. $\dot{A}_2$ is obtained from $A_1 \leftrightarrow A_2$. The self-interaction coefficient a and the "nonresonant" term b_0 are $\mathcal{O}(\gamma)$.

Of central interest in (5) is the contribution b_{res} capturing the effect of the slaved modes $Z_3^{\pm}$. Loosely speaking, if $b_{\text{res}} > 0$ the stability of patterns involving the angle θ_{res} is enhanced, while $b_{\text{res}} < 0$ is suppressing [12].

The most important damped modes are those for which b_{res} is $\mathcal{O}(\gamma)$ or larger and there are here two possibilities. If some of the Q_ℓ are $\mathcal{O}(1)$ then b_{res} is $\mathcal{O}(\gamma^{-1})$ and dominates in the weak damping limit; this happens only for $\Omega = m$ and is in essence a single-frequency phenomenon (the "first harmonic" resonance). If $\Omega \neq m$ and some of the Q_ℓ are $\mathcal{O}(\gamma)$ then b_{res} is $\mathcal{O}(\gamma)$, and hence comparable to b_0. This occurs when $\Omega \in \{2m, n, m \pm n, n - m\}$ for some forcing frequency component n. Each of these conditions on Ω generates a particular type of coupling at $\mathcal{O}(\gamma)$. For example, with $\Omega = n - m$ we have $F_2 = F_7 = f_n$. If more than one condition is satisfied, there are more coupling terms (e.g., if $\Omega = n - m = m - p$, then one would also have $F_1 = \bar{F}_5 = f_p$).

In addition to the issue of coupling terms, it matters whether or not the damped mode is parametrically forced at $\mathcal{O}(\gamma)$. This forcing is present when there is a frequency 2Ω in $F(t)$ and magnifies the resonance effect as it brings the damped mode closer to criticality. It also gives rise to an important phase dependence as the parametric forcing of $f_{2\Omega}$ competes with the nonlinear forcing of $\mathcal{Q}_3$ [see Eqs. (4)]. The ability to reveal phase information, via T_τ, is a crucial advantage of starting with a TW description.

In Table I we give the leading contribution to b_{res} of the important damped modes for forcing functions containing up to three frequencies (m, n, p). To simplify the expressions therein, we define

$$\alpha_1 = q_1 q_5, \qquad \alpha_2 = q_2 q_7, \qquad \alpha_3 = 2q_6(q_3 - q_4),$$

$$\alpha_4 = q_1 q_7 - q_2 q_5, \qquad \alpha_5 = \epsilon_\lambda(2q_1 q_6 + q_5(q_3 - q_4)),$$

$$\alpha_6 = \epsilon_\lambda(2q_2 q_6 - q_7(q_3 - q_4)), \qquad \epsilon_\lambda = \text{sign}(\lambda_i), \qquad (6)$$

and the functions

TABLE I. Leading resonant contribution b_{res} in (5) for the most important damped modes under appropriate choice of (up to) three-frequency forcing. Here, $m, n, p, \Omega > 0$ and $x \in \mathbb{Z}^+$. Each expression for (m, n, p), given Ω, is excluded from those of entries further down the table. Dots in the first column indicate an arbitrary commensurate frequency (if present). For $\star$ the $\pm$ follows sign $(m - n)$.

(m, n, p)	Ω	Leading resonant contribution b_{res}	Relevant phase(s)
$(m, \cdot, \cdot)$	m	$-\alpha_1/\lvert\varrho\rvert$	
$(m, \cdot, \cdot)$	$2m$	$-\alpha_1\lvert f_m\rvert^2/\lvert\varrho\rvert$	
$(m, n, \cdot)$	n	$-\alpha_3\lvert f_n\rvert^2/\lvert\varrho\rvert$	
$(m, n, \cdot)$	$m \pm n$	$-\alpha_1\lvert f_n\rvert^2/\lvert\varrho\rvert$	
$(m, n, \cdot)$	$n - m$	$\alpha_2\lvert f_n\rvert^2/\lvert\varrho\rvert$	
$(m, 2m, \cdot)$	m	$-\alpha_1 P_n(\Phi)$	$\Phi = \phi_n - 2\phi_m$
$(m, 4m, \cdot)$	$2m$	$-\alpha_1\lvert f_m\rvert^2 P_n(\Phi)$	$\Phi = \phi_n - 4\phi_m$
$(m, n, 2n)$	n	$-\alpha_3\lvert f_n\rvert^2 P_p(\Phi)$	$\Phi = 2\phi_n - \phi_p$
$(3x, 2x, \cdot)$	x	$-\alpha_1\lvert f_n\rvert^2 P_n(\Phi)$	$\Phi = 3\phi_n - 2\phi_m$
$(m, n, 2m \pm 2n)$	$m \pm n$	$-\alpha_1\lvert f_n\rvert^2 P_p(\Phi)$	$\Phi = \phi_p - 2\phi_m \mp 2\phi_n$
$(m, n, 2n - 2m)$	$n - m$	$\alpha_2\lvert f_n\rvert^2 P_p(\Phi)$	$\Phi = \phi_p + 2\phi_m - 2\phi_n$
$(m, 2m, \cdot)$	$2m$	$(-\alpha_1\lvert f_m\rvert^2 - \alpha_3\lvert f_n\rvert^2 + \alpha_5\lvert f_m\rvert\lvert f_n\rvert \sin\Phi)/\lvert\varrho\rvert$	$\Phi = \phi_n - 2\phi_m$
$(m, 3m, \cdot)$	$2m$	$(-\alpha_1\lvert f_m\rvert^2 + \alpha_2\lvert f_n\rvert^2 + \alpha_4\lvert f_m\rvert\lvert f_n\rvert \cos\Phi)/\lvert\varrho\rvert$	$\Phi = \phi_n - 3\phi_m$
$(m, n, \lvert m - n\rvert)$	n	$(-\alpha_1\lvert f_p\rvert^2 - \alpha_3\lvert f_n\rvert^2 + \alpha_5\lvert f_n\rvert\lvert f_p\rvert \sin\Phi)/\lvert\varrho\rvert$	$\Phi = \phi_n - \phi_m \pm \phi_p \star$
$(m, n, m + n)$	n	$(\alpha_2\lvert f_p\rvert^2 - \alpha_3\lvert f_n\rvert^2 + \alpha_6\lvert f_n\rvert\lvert f_p\rvert \sin\Phi)/\lvert\varrho\rvert$	$\Phi = \phi_m + \phi_n - \phi_p$
$(m, n, 2m \pm n)$	$m \pm n$	$(\alpha_2\lvert f_p\rvert^2 - \alpha_1\lvert f_n\rvert^2 + \alpha_4\lvert f_n\rvert\lvert f_p\rvert \cos\Phi)/\lvert\varrho\rvert$	$\Phi = 2\phi_m - \phi_p \pm \phi_n$
$(3, 1, 2)$	1	$-\alpha_1\lvert f_p\rvert^2 P_p(\Phi_1 - \Phi_2) - \alpha_3\lvert f_n\rvert^2 P_p(\Phi_1 + \Phi_2)$ $+\alpha_5\lvert f_n\rvert\lvert f_p\rvert R_p(\Phi_1, \Phi_2)$	$\Phi_1 = \phi_n - \phi_m + \phi_p$ $\Phi_2 = \phi_m + \phi_n - 2\phi_p$
$(3, 2, 4)$	1	$-\alpha_1\lvert f_n\rvert^2 P_n(\Phi_1 + \Phi_2) + \alpha_2\lvert f_p\rvert^2 P_n(\Phi_2 - \Phi_1)$ $+\alpha_4\lvert f_n\rvert\lvert f_p\rvert R_n(\Phi_1 - 90°, \Phi_2 + 90°)$	$\Phi_1 = \phi_n + \phi_p - 2\phi_m$ $\Phi_2 = 2\phi_n - \phi_p$
$(1, 2, 3)$	2	$(\alpha_2\lvert f_p\rvert^2 - \alpha_1\lvert f_m\rvert^2 - \alpha_3\lvert f_n\rvert^2 + \alpha_4\lvert f_m\rvert\lvert f_p\rvert \cos\Phi_1$ $+\alpha_5\lvert f_m\rvert\lvert f_n\rvert \sin\Phi_2 + \alpha_6\lvert f_n\rvert\lvert f_p\rvert \sin(\Phi_2 - \Phi_1))/\lvert\varrho\rvert$	$\Phi_1 = \phi_p - 3\phi_m$ $\Phi_2 = \phi_n - 2\phi_m$
$(1, 2, 4)$	2	$-\alpha_1\lvert f_m\rvert^2 P_p(\Phi_1 - \Phi_2) - \alpha_3\lvert f_n\rvert^2 P_p(\Phi_1 + \Phi_2)$ $+\alpha_5\lvert f_m\rvert\lvert f_n\rvert R_p(\Phi_1, \Phi_2)$	$\Phi_1 = \phi_n - 2\phi_m$ $\Phi_2 = \phi_n + 2\phi_m - \phi_p$
$(1, 3, 4)$	2	$-\alpha_1\lvert f_m\rvert^2 P_p(\Phi_1 - \Phi_2) + \alpha_2\lvert f_n\rvert^2 P_p(\Phi_1 + \Phi_2 + 180°)$ $+\alpha_4\lvert f_m\rvert\lvert f_n\rvert R_p(\Phi_1 + 90°, \Phi_2 + 90°)$	$\Phi_1 = \phi_n - 3\phi_m$ $\Phi_2 = \phi_m + \phi_n - \phi_p$

$$P_{2\Omega}(\Phi) = \frac{(|\varrho| + \mu_i|f_{2\Omega}|\sin\Phi)}{(|\varrho|^2 - |\mu_i f_{2\Omega}|^2)}, \quad (7)$$

$$R_{2\Omega}(\Phi_1, \Phi_2) = \frac{(|\varrho|\sin\Phi_1 + \mu_i|f_{2\Omega}|\cos\Phi_2)}{(|\varrho|^2 - |\mu_i f_{2\Omega}|^2)}, \quad (8)$$

where Φ, Φ_1, and Φ_2 are particular T_τ-invariant combinations of the ϕ_u.

There are four groupings in the table. The first shows the five important damped modes and their contribution to b_{res} when there is only one type of coupling at $\mathcal{O}(\gamma)$ or lower and no parametric forcing $f_{2\Omega}$. In these cases there is no (leading order) dependence on the forcing phases. In the second section the same damped modes have been parametrically forced. The factor $1/|\varrho|$ is then replaced by $P_{2\Omega}(\Phi)$ of (7), a strictly positive oscillatory function ($|\varrho| > |\mu_i f_{2\Omega}|$ for damped modes) with extrema at $\Phi = \pm 90°$. Entries in the third section display two types of coupling, while the final five cases in the table have $f_{2\Omega}$ forcing as well. Note that equivalent cases can be trivially generated from those in Table I by switching n and p and relabeling, for example $(m, n, p = n - m)$, $\Omega = p$ is equivalent to $(m, n, p = m + n)$, $\Omega = n$.

If the undamped problem has a Hamiltonian structure the results of Table I must be augmented. Specifically, if for $\gamma = 0$ Eqs. (4) derive from a Hamiltonian $\mathcal{H}$ through $dZ_j^{\mp}/dt = \mp i\partial\mathcal{H}/\partial\bar{Z}_j^{\pm}$ (see, e.g., [13]) then, allowing for simple rescalings of the dynamical variables, we have $\alpha_1 > 0$, $\alpha_2 > 0$, $\alpha_3 = 0$, $\alpha_4 = 0$. These relations mean that for simple couplings (the first two sections of Table I) the sign of b_{res} is determined, and hence its effect on patterns involving θ_{res}. The $\Omega = \{m, 2m, m \pm n\}$ modes are suppressing, the $\Omega = n$ mode is inconsequential, and the $\Omega = n - m$ mode is enhancing.

We pursue the implications of Table I by examining two cases relevant to recent experiments on multifrequency forced Faraday waves. We compare our theoretical predictions with coefficients calculated numerically from the Zhang-Viñals Faraday wave equations [9], which describe weakly damped fluids in deep containers. The calculation (see [12]) gives us the cross-coupling coefficient $b = b_0 + b_{res}$ at the SW bifurcation and is independent of the symmetry arguments used here.

First we consider the superlattice-I pattern observed with two-frequency $(m, n) = (6, 7)$ forcing in the experiments of [14]. The critical wave vectors comprising this

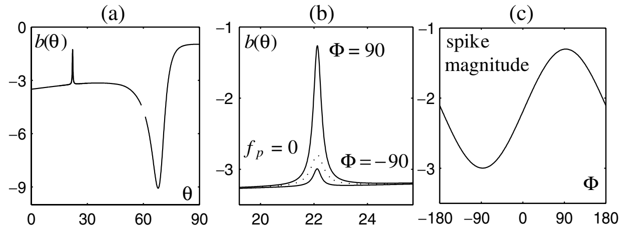

FIG. 2. Effect of Φ on the stabilizing $\Omega = n - m$ resonance that selects superlattice patterns [14] with $(m, n, p) = (6, 7, 2)$ forcing. (a) $b(\theta)$ with the optimal phase $\Phi = 90°$; the singular region heralding hexagons at $\theta = 60°$ is removed. The large dip near $\theta = 67°$ is due to the strongly suppressing $\Omega = m$ resonance in the first section of Table I. (b) Close-up of $b(\theta)$ near $\theta = 22°$ with $\Phi = 90°$ and $\Phi = -90°$; the two-frequency result (dotted line) with $f_p = 0$ is also shown. (c) Spike magnitude versus Φ [see Eq. (7)]. For these calculations $|f_n|/|f_m| = 0.75$, $|f_p|/|f_m| = 0.1$. The rescaled fluid parameters (see [10]) in the Zhang-Viñals equations are $\gamma = 0.08$, $G_0 = 1.5$.

pattern lie at the vertices of two hexagons, one rotated by an angle θ_h with respect to the other. It was shown in [12] that the experimentally observed angle of $\theta_h \simeq 22°$ is related to a resonant triad at $\theta \simeq 158°$ $(=180° - 22°)$ involving the $\Omega = n - m$ mode. This is the most intriguing of the damped modes because it gives $b_{res} > 0$ and thus acts as a selection mechanism. Table I indicates how to enhance this effect: parametrically force the damped mode using $(m, n, 2n - 2m) = (6, 7, 2)$ forcing and choose the correct phase. Figure 2(a) shows $b(\theta)$ for $\Phi = 90°$, which optimizes the stabilizing effect at $\theta_h \simeq 22°$. Figure 2(b) shows that parametrically forcing the $\Omega = n - m$ mode with this phase quadruples the stabilizing impact with respect to the two-frequency case used in the experiments. If the wrong phase $(\Phi = -90°)$ is chosen, the effect of the resonance will actually be diminished compared to the two-frequency case. Figure 2(c) shows the sinusoidal dependence of b_{res} on Φ, in excellent agreement with our predictions.

As a second example, we consider recent experimental results on quasipatterns. It was reported in [15] that eightfold quasipatterns, which were distorted and difficult to observe with $(m, n) = (3, 2)$ forcing, became dramatically cleaner and more robust with $(m, n, p) = (3, 2, 4)$ forcing. An explanation for this is provided by Table I. Specifically, from the dispersion relation, we find that the $\Omega = 1$ mode forms a resonant triad with the critical modes with associated angle $\theta_{res} \approx 43°$. This is nearly the angle present in the distorted (3,2)-forced quasipatterns in [15]. Table I indicates that with $(3, 2, 4)$ forcing there is a positive $\alpha_2|f_4|^2$ contribution to b_{res}. Numerical investigations reveal that the stabilizing spike in $b(\theta)$ becomes broader with increasing γ [16], making it reasonable to expect that for the relatively large damping of the experiments the stabilization extends to the 45° angle associated with the perfect eightfold quasipattern.

This Letter demonstrates how multifrequency forcing may be used to control certain resonant triad interactions relevant to pattern formation in spatially extended, parametrically forced systems. In general, the influence of damped, resonant modes depends on particular combination(s) of forcing phases. Using the "proper" phase greatly enhances resonance effects while the "wrong" phase can actually reduce them. Although we have gone to the limit of weak damping and forcing to obtain these results, we emphasize that there is still a preferred phase even for higher damping [16]. We applied our results to Faraday waves, where many experiments are available (see, e.g., [17]), but we expect the framework we developed here will be useful for studying and controlling other systems such as those in nonlinear optics [18].

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