Critical Dimension in the Black-String Phase Transition

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In spacetimes with compact dimensions, there exist several black object solutions including the black hole and the black string. They may become unstable depending on their relative size and the length scales in the compact dimensions. The transition between these solutions raises puzzles and addresses fundamental questions such as topology change, uniquenesses, and cosmic censorship. Here, we consider black strings wrapped over the compact circle of a d-dimensional cylindrical spacetime. We construct static nonuniform strings around the marginally stable uniform string. First, we compute the instability mass for a large range of dimensions and find that it follows an exponential law γ^d, where $\gamma < 1$ is a constant. Then we determine that there is a critical dimension, $d_* = 13$, such that for $d \leq d_*$ the phase transition is of first order, while for $d > d_*$ it is, surprisingly, of higher order.

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In $4d$, the static uncharged black hole (BH) solutions with a given mass are stable and unique. However, the fundamental theory of nature which, as now believed by many, is the string/M-theory, contains more than four dimensions. In this situation, the phase space of massive solutions of general relativity is much more rich and varied. Several phases of solutions exist and transitions between them may occur. For concreteness, we consider the background with a single compact dimension, i.e., with the topology of a cylinder, $\mathbb{R}^{d-2,1} \times \mathbf{S}^1$. The coordinate along the compact direction is denoted by z and its asymptotic length is L. The problem is characterized by a single dimensionless parameter:

$$\mu := G_d M / L^{d-3}, \tag{1}$$

where G_d is the d-dimensional gravitational constant and M is the mass.

Gregory and Laflamme (GL) [1,2] discovered that the uniform black string (i.e., a $d-1$ Schwarzschild solution times a circle, which is the large mass solution) develops a dynamical instability if the compactification radius is "too large." Their interpretation was that the string decays to a single localized BH. In this case, the horizon pinches off and the central singularity becomes "naked." By now there is a rapidly growing amount of the literature on the subject [3–21]. In particular, the scenario of GL was questioned by Horowitz and Maeda (HM) [3] who, on grounds of the classical "no tear" property of the horizons, argued that horizon pinching is impossible and, hence, a decaying string settles to another stable phase—a nonuniform black string (NUBS). However, (partial) evidence against that has come from Gubser [4] who in $5d$ studied perturbative NUBSs emerging from the GL point. He showed that such solutions are too massive and have too low an entropy to serve as an end state of a decaying critical string. Namely, the transition to this NUBS is of first order and it is again unclear what state is accessed by the classically decaying GL string. Wiseman [7] reached the same conclusion by constructing the NUBS solutions numerically in $6d$ in a fully nonlinear regime. (In $5d$, [12,13] could be regarded as additional circumstantial evidence *contra* the HM claim.) However, in this Letter we discover that the transition to NUBS can be smooth depending on d.

Generalizing Gubser's $5d$ procedure [4], which is a version of the "marginal stability" method, we construct numerically d-dimensional static perturbative NUBS solutions around the GL point. First, we note that the GL instability mass exhibits to good accuracy an exponential scaling with d. Moreover, we find that there is a critical dimension, $d_* = 13$, below which the uniform-nonuniform strings transition is of first order. That is, it is qualitatively similar to what Gubser has found in $5d$. However, above d_* the NUBS solutions emerging from the instability point have a lower mass and a larger entropy than those of the critical string. Namely, the transition between the phases can be continuous. (This is consistent with the prediction of a critical dimension $\hat{d} = 10$ at the "merger point" of this system, where the string and the BH branches merge [6].)

Hence, the NUBS state is accessible by an unstable uniform string. In this case, the horizon would not pinch off at the GL point. Our result suggests, however, that the horizon fragmentation during the classical decay can be avoided only for $d > d_*$. This is a rather curious development since the original HM argument was dimension independent. It should be noted, however, that the central issue of whether any unstable string must decay to a string remains unresolved even for $d > d_*$.

The most general ansatz for static black-string solutions is

$$ds^2 = -e^{2A} f dt^2 + e^{2B}(f^{-1} dr^2 + dz^2) + e^{2C} r^2 d\Omega^2_{d-3},$$
$$f = 1 - 1/r^{d-4}, \tag{2}$$

where A, B, and C depend on r, z only. When these

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functions vanish the metric becomes that of a static uniform black string with the horizon located at $r_0 = 1$.

Gubser [4] has considered static NUBS solutions that differ only perturbatively from a uniform black string. Since the method was described in detail in the original paper [4] and then in [7], we mention only the most important points. Gubser developed a perturbation theory considering the expansion of the metric functions in powers of $\hat{\lambda}$. This $\hat{\lambda}$ parametrizes the NUBS branch that joins the GL point in the limit $\hat{\lambda} \to 0$. The expansion has the following form [Gubser used the "nonuniformity" parameter, $\lambda := 1/2(R_{\max}/R_{\min} - 1)$, where $R_{\max}$ and $R_{\min}$ refer to the z-dependent Schwarzschild radius of the horizon. It was shown subsequently in [16,19] that a good order parameter that allows one to put black strings and holes on the same phase diagram is not λ, which is undefined for the latter, but the scalar charge of the dilatonic field. However, for our current purposes $\hat{\lambda}$ may be left unspecified.]:

$$X = \sum_{n=0}^{\infty} \hat{\lambda}^n X_n(r)\cos(nKz), \qquad X_n(r) = \sum_{p=0}^{\infty} \hat{\lambda}^{2p} X_{n,p}(r),$$
$$K = \sum_{q=0}^{\infty} \hat{\lambda}^{2q} k_q, \tag{3}$$

for $X = A, B, C$ with $X_{0,0} = 0$; and $K = 2\pi/L$.

Upon substituting (3) into the Einstein equations, $R_{\mu\nu} = 0$, a finite set of ordinary differential equations (ODEs) is generated at each order of the expansion. (See [17] for derivation of the Einstein equations in a similar case.) Gubser's method is very accurate up to the third order in $\hat{\lambda}$. Following the original procedure, we restrict our computations up to $O(\hat{\lambda}^3)$. Nevertheless, interesting results are already obtained here. Actually, the third order is precisely what one needs to determine the smoothness of the transition.

As discussed in [4], the perturbation theory contains a "scheme" dependence that seems to correspond to different parametrizations of the nonuniform branch. Originally, fixing of the "scheme" was achieved by fixing the constants $c_{n,p} := C_{n,p}(r_0)$. Still, other "schemes" can be used. For example, in [7] the asymptotic length of the compact circle was held fixed, $K = \text{const}$, but the constants $c_{n,p}$ were allowed to vary. In fact, different "schemes" all produce the same scheme-independent results, such as, e.g., the dimensionless mass (1). Here we choose to work in the "standard scheme," as it is referred in [4], by fixing $c_{n,p} = 0$ for $n > 1$ and $c_{1,0} = 1$.

Once the metric functions are known, various thermodynamical variables can be computed. Asymptotically, the spacetime (2) is characterized by two charges [16,19]—the mass and the tension of the black string. By making a Kaluza-Klein reduction in the z direction, $X_{n,p}$ in (3) are observed to be massive modes for $n > 0$ and they are massless otherwise. Only the latter contribute to the asymptotic charges since the former decay exponentially. Up to $O(\hat{\lambda}^3)$, the relevant massless modes are $X_{0,1}$. Asymptotically, they fall off as inverse powers of r. We denote the coefficients of the leading terms by X_∞. It is convenient to define the *variation* of the charges of a nonuniform string with respect to a uniform one. According to [16], at the leading order these variations read (we use units in which $G_N := G_d/L = 1$)

$$\delta M/M = -2[A_\infty + B_\infty/(d-3)]\hat{\lambda}^2,$$
$$\delta\mathcal{T}/\mathcal{T} = -2[A_\infty + (d-3)B_\infty]\hat{\lambda}^2. \tag{4}$$

We also compute the variation in the temperature, $\delta T/T = \exp[A - B] - 1$, and in the entropy, $\delta S/S = \exp[B - (d-3)C] - 1$, which are evaluated at $r = 1$.

Finally, defining the variation of K, $\delta K/K := (k_1/k_0)\hat{\lambda}^2$, we determine the dimensionless, scheme-independent variables by multiplying the dimensional quantities by suitable powers of K. By doing so, we obtain for our variables

$$\delta\mu/\mu = \delta M/M + (d-4)\delta K/K := \eta_1\hat{\lambda}^2 + \cdots,$$
$$\delta\tau/\tau = \delta\mathcal{T}/\mathcal{T} + (d-4)\delta K/K := \tau_1\hat{\lambda}^2 + \cdots,$$
$$\delta\theta/\theta = \delta T/T - \delta K/K := \theta_1\hat{\lambda}^2 + \cdots,$$
$$\delta s/s = \delta S/S + (d-3)\delta K/K := s_1\hat{\lambda}^2 + \cdots. \tag{5}$$

Incorporating the first law as in [4], we evaluate the entropy difference between the nonuniform and uniform strings with the *same* mass

$$\frac{S_{\text{nonuniform}}}{S_{\text{uniform}}} = 1 + \sigma_1\hat{\lambda}^2 + \sigma_2\hat{\lambda}^4 + \cdots,$$
$$\sigma_1 = \eta_1 - \frac{d-4}{d-3}s_1,$$
$$\sigma_2 = -\frac{d-3}{2(d-4)}\left(\theta_1 + \frac{1}{d-4}\eta_1\right)\eta_1. \tag{6}$$

The vanishing of σ_1 is ensured by the first law at the leading order (where $L = \text{const}$) [4]. We verified that to a good ($\lesssim 1\%$) accuracy, $\sigma_1 \approx 0$ for our solutions. Thus, the entropy difference (6) arises only at $O(\hat{\lambda}^4)$.

At each order of $\hat{\lambda}$, we solved the ODEs numerically [22]. We were able to exactly reproduce the numbers found thus far in the literature: for $5d$ in [4] and for $6d$ in [7]. An indication of the accuracy of the method is gained by varying the "scheme" [4], by altering $c_{0,1} = 0, \pm 1$. The resulting variation in (5) and (6) gives an idea of the numerical uncertainty. For small d, the accuracy of our calculation is high, being about 0.5% in η_1 and 1% in σ_2. For larger d, the method is somewhat less accurate, yielding 5% and 6% variations in η_1 and σ_2, respectively, for $d = 16$. This has to do with the steep asymptotic falloff of A and B in which the leading terms decay as $r^{-(d-4)}$, while C falls off only as $1/r$ [in $5d$ the falloff is $\log(r)/r$]. Hence, the accuracy in extracting the coefficients A_∞, B_∞, that contribute to η_1 and σ_2, decreases for large d.

The critical mass.—The calculation in the linear order in $\hat{\lambda}$ yields the mass of the critical string, since the leading order of (3) corresponds to the static GL mode. We performed the calculations in $d = 5, \ldots, 16, 20, 30$, and 50. For $d \leq 10$, we confirm a very good agreement with the original GL results [1], presented in their Fig. 1. Note, however, that the methods are very different. For the entire range of d, we find that the critical mass is remarkably well approximated by

$$\mu_c \propto \gamma^d, \tag{7}$$

with $\gamma \simeq 0.686$, and the prefactor is approximately 0.47, for our definition of mass (1). In Fig. 1, we plot the relative *difference* between the logarithm of the critical mass and the fit (7). It is clearly seen that $\log(\mu_c)$ is linear for all d. There is still room for a weak d dependence, of order 2.1%, around the dominant scaling (7). We, however, could not extract this residual dependence.

To get an insight into this behavior (7), we compute the mass of a uniform black string whose entropy is equal to that of a single BH with the same mass. First, we compare the entropy of the black string with that of a d-dimensional Schwarzschild BH. Equating, $S_{\rm BH}^{(0)}(\mu) = S_{\rm BStr}(\mu)$, we solve for the mass:

$$\mu^{(0)} = \frac{1}{16\pi}\frac{\Omega_{d-3}^{d-3}}{\Omega_{d-2}^{d-4}}\frac{(d-3)^{(d-3)(d-3)}}{(d-2)^{(d-2)(d-4)}}, \tag{8}$$

where Ω_d is the surface area of a unit $\mathbf{S}^d$ sphere.

Actually, we can do slightly better by using the analytical formula for the entropy of *small* BHs on cylinders derived recently in [21]:

$$S_{\rm BH}^{(1)} = S_{\rm BH}^{(0)}\left[1 + \frac{\zeta(d-3)16\pi\mu}{2(d-3)\Omega_{d-2}} + O(\mu^2)\right], \tag{9}$$

where $\zeta(x)$ is Riemann's zeta function. This formula reflects the leading order corrections to the Schwarzschild metric due to compactification. (The perturbation theory is constructed in powers of $\mu \ll 1$.) It implies that for a given mass the entropy of a "caged black hole" (a BH in a compactified spacetime) is larger than the entropy of a Schwarzschild BH. The mass $\mu^{(1)}$ corresponding to equality of the entropies is then obtained by solving the equation $S_{\rm BH}^{(1)}(\mu) = S_{\rm BStr}(\mu)$.

We add to Fig. 1 the plots for these masses. In contrast to $\log(\mu_c)$, the logarithms of $\mu^{(0)}$ and $\mu^{(1)}$ have a nonlinear behavior for small d. They do, however, become linear (with different slopes) for $d \gg 10$. Here we already see a hint of a critical dimension—looking at the difference between μ_c and its estimator (either $\mu^{(0)}$ or $\mu^{(1)}$) one notices a change of sign at about $d \sim 12.5$. This suggest that for $d \gtrsim 13$ the BH state is entropically favorable over the string state only for $\mu < \mu_c$.

From a sudden to a smooth phase transition.— Performing the computation in higher orders, up to $O(\hat{\lambda}^3)$, we obtain the variation in the variables (5) and entropy (6). The results for η_1 and σ_2 are depicted in Fig. 2. One observes that η_1 is initially positive for $d = 5$, reaches a maximum at $d = 10$, and becomes negative for $d > 13$. Then it continues to decrease and in fact it drops increasingly faster with d, as indicated by the growing distances between subsequent points in the graph. The pattern for σ_2 is similar but with the opposite sign. (In fact we also did the computation in $d = 20$ finding the same trends. However, the numerical errors were of order 20% so we regard this case as indicative only.)

The key phenomena is the appearance of a critical dimension, $d_* = 13$, above which the perturbative nonuniform strings are less massive than the marginal GL string. Moreover, their entropy is larger than the entropy of the uniform string with the same mass. It is important that η_1 and σ_2 change signs simultaneously.

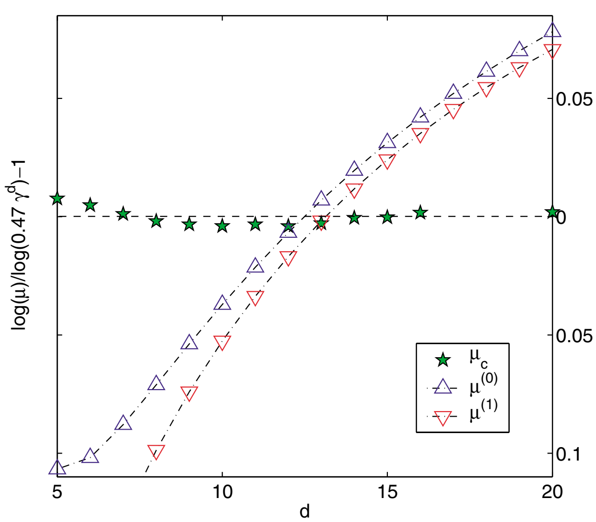

FIG. 1 (color online). The relative *difference* between the mass and the fit (7), $0.47\gamma^d$, as a function of d. For μ_c this difference is zero with the spread of about 0.8% magnitude, giving approximately 2.1% variations in μ_c itself.

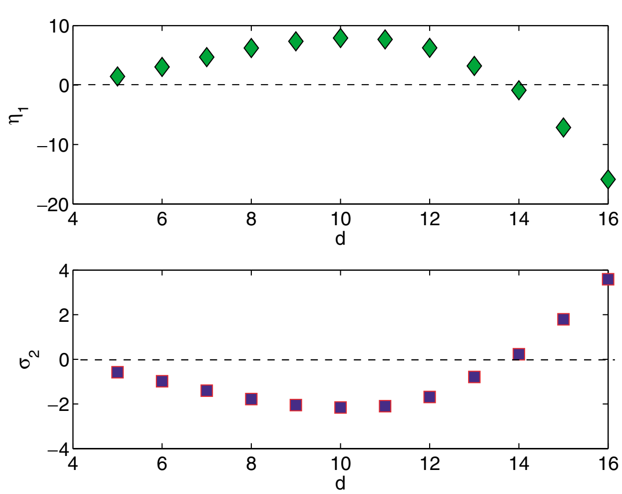

FIG. 2 (color online). The trends in the mass, $\mu_{\rm nonuniform}/\mu_{\rm uniform} := 1 + \eta_1\hat{\lambda}^2 + \cdots$, and the entropy, $S_{\rm nonuniform}/S_{\rm uniform} := 1 + \sigma_2\hat{\lambda}^4 + \cdots$, shifts between uniform and nonuniform black strings. The key result is the sign change of η_1 and σ_2 above $d_* = 13$.

As for the other variables, we find that the trend in the entropy shift, s_1, is qualitatively similar to the behavior of η_1—it is positive for $d \leq d_*$ and it becomes negative above d_*. For the variation of the temperature, we note that below d_* the NUBS is "cooler" than the uniform one, and above d_* it is "hotter." We find that the tension of the nonuniform strings is lower than that of uniform ones. This is in tune with the expectation that the uniform black string has a maximal tension, and that the tension vanishes for small black holes [16,23]. In addition, we observe the ratios η_1/τ_1 and η_1/s_1 to be discontinuous near d_*. Note also that in Fig. 2 we plot the coefficients of the mass and the entropy shifts. To obtain the physical variations, these and other coefficients must be multiplied by suitable powers of $\hat{\lambda}$.

In summary, while we have found the dependence of the critical mass on the dimension, we do not have at present an explanation for the scaling (7). We believe it gives us some insight into the nature of the GL instability and it probably is connected with the thermodynamical instability of the system [14]. However, it is the appearance of a critical dimension, d_*, that can perhaps be regarded as our main result. It implies that above d_* the critical string can smoothly evolve into the NUBS phase. For $d \leq d_*$, the transition is of first order.

The continuous transition above d_* suggests that the NUBS phase can be a natural end state of the GL instability. Indeed, a uniform string losing its mass by evaporation and encountering the instability at μ_c can smoothly evolve to the nonuniform state keeping its singularity covered by the horizon. Already from Fig. 1 it could be inferred that above $d \gtrsim 13$ there can be a branch of solutions between the uniform strings and the BHs. We believe that the NUBS state is a reasonable candidate for this "missing link."

As the mass is further radiated away, two scenarios may be proposed: (i) The NUBS branch extends to an arbitrary small mass. A black string evolves along this branch probably increasing its nonuniformity all the way down to zero mass. In this case, the cosmic censorship would be held (at least until the final stages of evaporation). (ii) A NUBS becomes unstable at a finite mass where the horizon fragments and a localized BH forms. This may lead to a compromise of the cosmic censorship, much like in the $d \leq d_*$ case but for a mass smaller than μ_c. The transition between a NUBS and a BH can be sudden or smooth depending on the relative values of the instability masses for these states. Note that a NUBS branch that extends to zero mass or becomes unstable even earlier on a phase diagram is conceptually the same. The main difference is whether the naked singularity shows up before the end of evaporation or not.

To address these intriguing issues, it would be a very interesting future task to construct in a fully nonlinear regime, as in [7], the branch of NUBSs that we found here. In particular, it is interesting to determine for how low a mass this branch drops, would the horizon try to pinch off forming a conelike "waist" [6,9], and whether the topology tends to change. In addition, we expect that a time evolution of the critical string, as in [12], should confirm a nice decay for $d > d_*$.

In this work we have considered black strings in a cylindrical spacetime. We believe that the critical dimension phenomena is general and will hold for more general backgrounds with additional compact dimensions even if the specific value $d_* = 13$ would change.

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