Particle Beams Guided by Electromagnetic Vortices: New Solutions of the Lorentz, Schrödinger, Klein-Gordon, and Dirac Equations

Iwo Bialynicki-Birula*

Center for Theoretical Physics, Polish Academy of Sciences, Al. Lotniko´w 32/46, 02-668 Warsaw, Poland (Received 11 February 2004; revised manuscript received 15 March 2004; published 9 July 2004)

It is shown that electromagnetic vortices can act as beam guides for charged particles. The confinement in the transverse directions is due to the rotation of the electric and magnetic fields around the vortex line. A large class of exact solutions describing various types of relativistic beams formed by an electromagnetic wave with a simple vortex line is found both in the classical and in the quantum case. In the second case, the motion in the transverse direction is fully quantized. Particle trajectories trapped by a vortex are very similar to those in a helical undulator.

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Electromagnetic waves with vortices have been extensively studied both theoretically and experimentally. This field of research has became known as singular optics [1]. In this work I take these studies one step further and analyze the motion of charged particles in the vicinity of a vortex line. I shall consider the simplest possible solution of Maxwell equations with a straight vortex line and show that this configuration of the electromagnetic field acts as a perfect beam guide for charged particles. I study these nonspreading beams in the classical case, when the relativistic particle trajectory is determined by the Lorentz equations, and also in the quantum case, when the wave function describing the beam obeys the Schrödinger, Klein-Gordon, or the Dirac equation. In the classical and in the quantum case, I exhibit analytic solutions that enable one to fully understand the intricate dynamics of these beams.

The electric and magnetic field vectors of my model Maxwell field are

$$
E(x, y, z, t) = B_0 \omega(f(x, y, z, t), g(x, y, z, t), 0),
$$
 (1a)

$$
\mathbf{B}(x, y, z, t) = B_0 k \left(-g(x, y, z, t), f(x, y, z, t), 0 \right), \quad (1b)
$$

where B_0 is the field amplitude measured in units of the magnetic field and

$$
f(x, y, z, t) = x \cos(\omega t_-) + y \sin(\omega t_-), \tag{2a}
$$

$$
g(x, y, z, t) = x \sin(\omega t_-) - y \cos(\omega t_-), \tag{2b}
$$

where $t = t - z/c$. This configuration of the field is the simplest example of the EM field with a vortex line [2,3]. The solution of the Maxwell equations given by Eqs. (1) is not as artificial as it may look at a first glance. It is a fairly good approximation (near the *z* axis and not far from the waist compared to the Raleigh range) to a realistic circularly polarized paraxial Laguerre-Gauss beam with $n = 0$ and $m = 1$.

The Lorentz equations of motion

$$
m\ddot{\xi}^{\mu}(\tau) = e f^{\mu\nu}[\xi(\tau)] \dot{\xi}_{\nu}(\tau), \tag{3}
$$

for a particle moving in the field (1), expressed in terms of the components $(\xi, \eta, \zeta, \theta)$ of the four-vector $\xi^{\mu}(\tau)$, have the form (for the sake of brevity, I shall occasionally drop the dependence on the proper time τ)

$$
\ddot{\xi} = \omega_c \omega f(\xi, \eta, \zeta, \theta) (\dot{\theta} - \dot{\zeta}/c), \tag{4a}
$$

$$
\ddot{\eta} = \omega_c \omega g(\xi, \eta, \zeta, \theta) (\dot{\theta} - \dot{\zeta}/c), \tag{4b}
$$

$$
\ddot{\zeta} = \frac{\omega_c \omega}{c} [\dot{\xi} f(\xi, \eta, \zeta, \theta) + \dot{\eta} g(\xi, \eta, \zeta, \theta)], \qquad (4c)
$$

$$
c\ddot{\theta} = \frac{\omega_c \omega}{c} [\dot{\xi} f(\xi, \eta, \zeta, \theta) + \dot{\eta} g(\xi, \eta, \zeta, \theta)], \qquad (4d)
$$

where the dots denote derivatives with respect to τ and $\omega_c = eB_0/m$ is the cyclotron frequency. These equations are nonlinear but they can be explicitly solved owing to conservation laws.

By subtracting Eq. (4c) from Eq. (4d), one obtains $\ddot{\theta}$ – $\ddot{\zeta}/c = 0$ and this leads to the first conserved quantity

$$
\dot{\theta} - \dot{\zeta}/c = \sqrt{1 + (\dot{\xi}^2 + \dot{\eta}^2 + \dot{\zeta}^2)/c^2} - \dot{\zeta}/c = \mathcal{E}
$$

= const₁. (5)

Apart from the factor mc^2 , this constant is the light-front energy—the conjugate variable to $t_+ = t + z/c$

$$
\mathcal{E} = \frac{1 - v_z/c}{\sqrt{1 - v^2/c^2}} = \frac{\sqrt{m^2c^4 + p^2c^2} - p_zc}{mc^2}.
$$
 (6)

Without any loss of generality one may assume that $\theta(0) = 0 = \zeta(0)$ and then Eq. (5) integrated with respect to τ yields

$$
\theta - \zeta/c = \mathcal{E}\,\tau. \tag{7}
$$

Thus, in this case, the proper time is proportional to the light-front variable. The second constant of motion is obtained by combining Eqs. (4a)–(4c) and it reads

$$
\dot{\zeta} - \frac{1}{2c\mathcal{E}} (\dot{\xi}^2 + \dot{\eta}^2) = \frac{c}{2} \left(\frac{1}{\mathcal{E}} - \mathcal{E} \right) = \text{const}_2. \tag{8}
$$

Since the phase of the wave field changes in proper time

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with frequency $\omega \mathcal{E}$, I shall incorporate \mathcal{E} into ω and define the effective frequency $\Omega = \mathcal{E}\omega$.

Owing to Eq. (5), the transverse motion separates from the longitudinal motion and the equations for ξ and η may be solved first. This task is made easier by transforming Eqs. (4a) and (4b) to the frame rotating (in proper time) with the angular velocity $\Omega/2$ around the *z* axis which amounts to replacing ξ and η by the new variables

$$
\alpha(\tau) = \xi(\tau)\cos(\Omega\tau/2) + \eta(\tau)\sin(\Omega\tau/2), \tag{9a}
$$

$$
\beta(\tau) = -\xi(\tau)\sin(\Omega \tau/2) + \eta(\tau)\cos(\Omega \tau/2). \tag{9b}
$$

The equations of motion for α and β read

$$
\ddot{\alpha} = \Omega \dot{\beta} + (\Omega^2/4) \alpha + \omega_c \Omega \alpha, \qquad (10a)
$$

$$
\ddot{\beta} = -\Omega \dot{\alpha} + (\Omega^2/4) \beta - \omega_c \Omega \beta. \tag{10b}
$$

These equations result from the following Hamiltonian:

$$
H = \frac{1}{2m}(p_{\alpha}^2 + p_{\beta}^2) + \frac{m\omega_c^2}{2}(\alpha^2 + \beta^2)
$$

$$
-\left(\omega_c + \frac{\Omega}{2}\right)\alpha p_{\beta} - \left(\omega_c - \frac{\Omega}{2}\right)\beta p_{\alpha}.
$$
 (11)

Despite the quadratic form of the Hamiltonian, it is not exactly a harmonic oscillator—the frequencies of the oscillations depend through $\mathcal E$ on the initial conditions. Still, this Hamiltonian can be expressed in terms of the complex eigenmode amplitudes a_{\pm} and a_{\pm}^* (classical counterparts of the annihilation and creation operators),

$$
H = \Omega_+ a_+^* a_+ - \Omega_- a_-^* a_-, \tag{12}
$$

where $\Omega_{\pm} = \Omega \sqrt{1 \pm \kappa/2}$ and $\kappa = 4\omega_c/\Omega$ is a dimensionless parameter that controls the particle behavior in the *xy* plane. The amplitudes a_{\pm} have the form

$$
a_{+} = \frac{1}{2} \sqrt{\frac{\kappa_{+}}{\gamma}} \left(p_{\beta} - \gamma \alpha - i \frac{p_{\alpha} + \gamma \beta}{\kappa_{+}} \right), \tag{13a}
$$

$$
a_{-} = \frac{1}{2} \sqrt{\frac{\kappa_{-}}{\gamma}} \bigg[\frac{p_{\beta} + \gamma \alpha}{\kappa_{-}} + i (p_{\alpha} - \gamma \beta) \bigg], \tag{13b}
$$

where $\gamma = |eB_0|$ and $\kappa_{\pm} = \sqrt{1 \pm \kappa}$. The minus sign in the diagonal form of the Hamiltonian (12) indicates that the beam dynamics in the transverse plane is governed by the same combination of the attractive/repulsive oscillator forces and the Coriolis force as one encounters for a particle in the Paul trap [4], an electron Trojan wave packet in an atom [cf. Eq. (7) of Ref. [5]] or Trojan asteroids in the Sun-Jupiter system [5].

The general solution for $\xi(\tau)$ and $\eta(\tau)$ is obtained by solving Eqs. (10) in terms of eigenmodes and then undoing the rotation (9). The final expression for the motion of particles in the plane perpendicular to the vortex line can be compactly written in the complex form

$$
\xi(\tau) + i\eta(\tau) = e^{i\Omega\tau/2}[(iD\kappa_+ + A)\sin(\Omega_+\tau) - (B\kappa_- + iC)\sin(\Omega_-\tau) + (iA\kappa_+ - D)\cos(\Omega_+\tau) + (C\kappa_- - iB)\cos(\Omega_-\tau)],\tag{14}
$$

where the constants *A*, *B*, *C*, *D* depend on the initial values of the transverse positions and velocities

$$
A = (\eta_0 + \dot{\xi}_0 / 2\omega_c) / \kappa_+, \qquad B = \dot{\xi}_0 / 2\omega_c, \qquad (15a)
$$

$$
C = (\xi_0 + \dot{\eta}_0/2\omega_c)/\kappa_-, \qquad D = \dot{\eta}_0/2\omega_c.
$$
 (15b)

For $|\kappa|$ < 1 one obtains bounded oscillations around the

vortex line with four characteristic frequencies: $\Omega_{+} \pm$ $\Omega/2$ and $\Omega_{-} \pm \Omega/2$ and for $|\kappa| > 1$ one has runaway solutions with exponential growth. The motion along the *z* axis is obtained from Eq. (8) by a straightforward integration. The resulting formula for $\zeta(\tau)$ has two parts—a part with oscillating terms and a linear part in τ .

$$
\zeta(\tau) = \frac{\kappa^2 \omega (D^2 - A^2) \sin(2\Omega_+ \tau)}{16c\kappa_+} + \frac{\kappa^2 \omega (B^2 - C^2) \sin(2\Omega_- \tau)}{16c\kappa_-} - \frac{\kappa^2 \omega AD[1 - \cos(2\Omega_+ \tau)]}{8c\kappa_+} + \frac{\kappa^2 \omega CB[1 - \cos(2\Omega_- \tau)]}{8c\kappa_-} + \frac{c\tau}{2} \left[\frac{1}{\mathcal{E}} - \mathcal{E} + \frac{\mathcal{E}\kappa^2 \omega^2}{8c^2} (A^2 + B^2 + C^2 + D^2) \right].
$$
 (16)

Depending on the sign of the linear term, the guiding center of the beam may follow the electromagnetic wave or move in the opposite direction. By a special choice of initial conditions one may even get rid of the linear term altogether in which case the longitudinal motion will also be bounded, but it requires fine tuning. This complex behavior is a purely relativistic effect. In the nonrelativistic limit, the motion in the *z* direction is free, not affected by the wave at all, $\zeta(t) = v_z t$. In Fig. 1, I show two trajectories of electrons for different initial conditions. These trajectories are very similar to those in a helical undulator (an arrangement of permanent magnets

used to produce circularly polarized radiation). In the present case, the role of permanent magnets is played by an electromagnetic wave with a vortex line and the beam confinement is due to a totally different (Trojan) mechanism.

I shall start the analysis of the quantum-mechanical problem with the Klein-Gordon (KG) equation. The EM field (1) may be derived from the vector potential

$$
A(x, y, z, t) = B_0[-g(x, y, z, t), f(x, y, z, t), 0].
$$
 (17)

As seen from the analysis of the classical solutions, it is

FIG. 1. Two trajectories of electrons injected into the wave field (1) with $B_0 = 10^{-3}$ T and $\omega = 2\pi \times 10^9$ s⁻¹. The initial longitudinal momentum p_z of the electron is in both cases $25 \text{ keV}/c$ but they have different transverse momenta. The narrow trajectory has $p_x = 5 \text{ keV}/c$ while the wide one (wiggles on this trajectory are real) has $p_x = 50 \,\text{keV}/c$. The size of the box measured in wavelengths $2\pi c/\omega$ is $1\frac{1}{2} \times 1\frac{1}{2} \times 2\frac{1}{2}$.

preferable to use the coordinates *x*, *y*, and t_{\pm} . The KG equation in these coordinates reads

$$
\frac{4}{c^2}\partial_+\partial_-\psi = \left(\Delta_\perp - \frac{e^2}{\hbar^2}A^2 - 2i\frac{e}{\hbar}A\cdot\nabla - \frac{m^2c^2}{\hbar^2}\right)\psi,
$$
\n(18)

where $\partial_{\pm} = \partial/\partial t_{\pm}$. Since the variable t_{+} does not appear in this equation, one may seek its solutions in the form

$$
\psi(x, y, t_-, t_+) = e^{-ic^2(Mt_+ + m^2t_-/M)/2\hbar} \tilde{\psi}(x, y, t_-). \tag{19}
$$

An additional phase factor, dependent on $t_$, has been introduced to remove the mass term. The essential dependence on t_{-} is still contained in the wave function ψ . The function $\tilde{\psi}(x, y, t_{-})$ obeys the following equation:

$$
i\hbar \partial_- \tilde{\psi} = \left[-\frac{\hbar^2}{2M} \Delta_\perp + \frac{M \Omega_c^2}{2} (x^2 + y^2) \right] \tilde{\psi} + i\hbar \Omega_c [(x \partial_y + y \partial_x) \cos(\omega t_-) - (x \partial_x - y \partial_y) \sin(\omega t_-)] \tilde{\psi}, \quad (20)
$$

where Δ_{\perp} is the transverse part of the Laplacian and $\Omega_c = eB_0/M$. This equation is *exactly the same* as a nonrelativistic Schrödinger equation except that the role of the mass *m* is played by the separation constant *M* and the time parameter is replaced by the light-front variable *t*. Therefore, everything that one can say about the solutions of the Eq. (20) applies to the solutions of the Schrödinger equation. Upon transforming Eq. (20) to a comoving frame by the substitution

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$$
\tilde{\psi} = \exp\bigg[-\frac{\omega t_-}{2}(x\partial_y - y\partial_x)\bigg]\phi, \tag{21}
$$

one finally obtains

$$
i\hbar\partial_{-}\phi = \left[-\frac{\hbar^2}{2M}\Delta_{\perp} + \frac{M\Omega_c^2}{2}(x^2 + y^2) \right] \phi + i\hbar [(\Omega_c + \omega/2)x\partial_y + (\Omega_c - \omega/2)y\partial_x] \phi.
$$
\n(22)

By rearranging the terms, one may establish that the particle in this frame moves effectively under the influence of the constant magnetic field $\mathbf{B} = (0, 0, M\omega/e)$ and an additional repulsive quadratic potential $V =$ $-(M\omega^2/8)(\kappa_+^2 x^2 + \kappa_-^2 y^2).$

All stationary solutions of Eq. (22) are most easily classified with the use of the creation and annihilation operators. These operators diagonalize the Hamiltonian

$$
\hat{H} = (\hat{p}_x^2 + \hat{p}_y^2)/2M + M\Omega_c^2(\hat{x}^2 + \hat{y}^2)/2 \n- (\Omega_c + \omega/2)\hat{x}\hat{p}_y - (\Omega_c - \omega/2)\hat{y}\hat{p}_x
$$
\n(23)

and are obtained from the classical amplitudes (13) by the replacements

$$
a_{\pm} \to \sqrt{\hbar} \hat{a}_{\pm}, \qquad a_{\pm}^* \to \sqrt{\hbar} \hat{a}_{\pm}^{\dagger}, \tag{24a}
$$

$$
(\alpha, \beta, p_{\alpha}, p_{\beta}) \rightarrow (\hat{x}, \hat{y}, \hat{p}_x, \hat{p}_y). \tag{24b}
$$

This leads to the following form of the Hamiltonian:

$$
\hat{H} = \frac{\hbar \omega}{2} \bigg[\kappa_+ \bigg(\hat{a}_+^* \hat{a}_+ + \frac{1}{2} \bigg) - \kappa_- \bigg(\hat{a}_-^* \hat{a}_- + \frac{1}{2} \bigg) \bigg]. \quad (25)
$$

Thus, in contrast to the Volkov solution in the plane wave EM field [6], the motion in the transverse direction is fully quantized. In contrast to the motion in a constant magnetic field, the particle is localized near the *z* axis. Different normalization of the classical and quantum Hamiltonian is due to the fact that the first one generates the evolution in proper time, while the second one generates the evolution in the $t_$ variable. These two parameters differ by the scaling factor \mathcal{E} . The quantum theory becomes consistent with the classical one when M/m is identified with $\mathcal E$. It means that $M/m = \Omega/\omega$ and, as a result, the value of κ encountered in quantum theory becomes equal to the classical one $(4\omega_c/\Omega = 4\Omega_c/\omega)$, as it should be.

Having diagonalized the Hamiltonian, one may generate the whole Fock space of stationary solutions. They are obtained by acting on the fundamental state ϕ_0 with the creation operators. The fundamental state is the one annihilated by both operators \hat{a}_{\pm} . Solving two simple differential equations $\hat{a}_{\pm} \phi_0 = 0$, one obtains

$$
\phi_0(x, y) = N \exp(-x^2/d_+^2 - y^2/d_-^2 - ixy/d^2), \quad (26)
$$

where the parameters d_{\pm} and *d* are given by

$$
d_{\pm}^{2} = \frac{\hbar}{\gamma} \frac{1 + \kappa_{+} \kappa_{-}}{\kappa_{\pm}}, \qquad d^{2} = \frac{\hbar}{\gamma} \frac{1 + \kappa_{+} \kappa_{-}}{1 - \kappa_{+} \kappa_{-}}.
$$
 (27)

The wave functions of the Fock states are polynomials in *x* and *y* multiplied by the Gaussian (26). In the laboratory frame these solutions are not stationary since the beams do not exhibit rotational symmetry around the *z* axis $(d_+ \neq d_-)$. In particular, the fundamental solution takes on a form of a rotating helix.

There is also a plethora of nonstationary solutions of Eq. (22). First, there are those that correspond directly to classical trajectories—the analogs of coherent states. The fundamental solution (26) corresponds to a trajectory which just sits on the vortex line, but one may easily obtain solutions of the KG equation representing *all other* classical trajectories. According to a general scheme [7] valid for all quadratic Hamiltonians, displacing any solution of the KG equation by the solutions of the classical equations of motion leads to new solutions. Applying such displacements to the solution (26), one obtains

$$
\phi(x, y, t_{-}) = N(\tau) e^{i[x p_{\alpha}(\tau) + y p_{\beta}(\tau)]/\hbar} \times \phi_0[x - \alpha(\tau), y - \beta(\tau)], \qquad (28)
$$

where $\tau = t_-/\mathcal{E}$ and the center-of-mass trajectories are obtained by solving the Hamilton's equations of motion that follow from (11). The time-dependent phase of the normalization constant is equal to the classical action [7]. To obtain the solution of the original equation, one must transform the wave function from the comoving frame back to the laboratory frame applying the inverse transformation to (21). Only then one obtains the quantummechanical counterparts of the classical trajectories.

Solutions based on a rigid Gaussian—the analogs of coherent states— do not exhaust all possibilities. Since the center of mass motion of the Gaussian wave packet decouples from its internal motion, one may easily generate solutions based on oscillating Gaussians—the analogs of squeezed states. The Gaussian parameters d_{\pm} and *d* for such states are functions of t . These states do not have direct classical counterparts and their complete analysis will be given elsewhere.

The solution of the Dirac equation proceeds along similar lines. I begin with rewriting the Dirac equation in the electromagnetic field (1)

$$
i\hbar \partial_t \Psi = [c\mathbf{\alpha} \cdot (-i\hbar \mathbf{\nabla} - e\mathbf{A}) + \beta mc^2] \Psi, \qquad (29)
$$

as a set of two coupled equations for the two-component wave functions

$$
2i\hbar\partial_+\Psi_+ = c[mc\sigma_z - \sigma_\perp \cdot (i\hbar\nabla + eA)]\Psi_-, \qquad (30a)
$$

$$
2i\hbar\partial_{-}\Psi_{-}=c[mc\sigma_{z}-\sigma_{\perp}\cdot(i\hbar\nabla+eA)]\Psi_{+},\qquad(30b)
$$

obtained with the use of the projections $P_{\pm} = (1 \pm \alpha_z)/2$

$$
\Psi = (P_+ + P_-)\Psi = \Psi_+ + \Psi_-\tag{31}
$$

The dependence on the variable t_{+} can again be separated by the same substitution (21), leading to

$$
Mc\tilde{\Psi}_+ = [mc\sigma_z - \sigma_\perp \cdot (i\hbar \nabla + e\mathbf{A})]\tilde{\Psi}_-, \qquad (32)
$$

$$
\left(2i\hbar\partial_{-} + \frac{m^2c^2}{M}\right)\tilde{\Psi}_{-} = c[mc\sigma_z - \sigma_{\perp}\cdot(i\hbar\nabla + eA)]\tilde{\Psi}_{+}.
$$
\n(33)

The first equation enables one to express $\tilde{\Psi}_+$ in terms of Ψ ₋ and leads to Eq. (20) for Ψ ₋. Again, as in the case of the KG equation, the dependence of the potential on *t* may be eliminated by the substitution (19) and the equation for Φ ₋ can be reduced to *the same* Eq. (22) as for a spinless particle. Still, the spin does play a role in the Dirac particle dynamics. Since the transformation to the comoving frame should also involve the spin part, the proper transformation rule, instead of (21), is now

$$
\tilde{\Psi}_{\pm} = \exp\bigg[-\frac{\omega t_{-}}{2}(x\partial_{y} - y\partial_{x} + i\sigma_{z}/2)\bigg]\Phi_{\pm}.
$$
 (34)

Finally, the wave equation for Φ in the comoving frame

$$
i\hbar \partial_{-} \Phi_{-} = \left[-\frac{\hbar^2}{2M} \Delta_{\perp} + \frac{M \Omega_{c}^2}{2} (x^2 + y^2) - \frac{\hbar \omega}{4} \sigma_{z} \right] \Phi_{-} + i\hbar \left[\left(\Omega_{c} + \frac{\omega}{2} \right) x \partial_{y} + \left(\Omega_{c} - \frac{\omega}{2} \right) y \partial_{x} \right] \Phi_{-} \tag{35}
$$

differs from Eq. (22) only by a simple spin term. Everything that has been said before about stationary solutions of the KG equation applies with almost trivial changes to the Dirac equation.

There are two properties of the solutions of the wave equations described here that might lead to new effects: the quantization of the transverse motion and the breaking of the rotational symmetry. This may help to observe the rotational frequency shift predicted some time ago [8] that depends crucially on these features.

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*Electronic address: birula@cft.edu.pl

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