Semiclassical Foundation of Universality in Quantum Chaos

Sebastian Müller,[1] Stefan Heusler,[1] Petr Braun,[1,2] Fritz Haake,[1] and Alexander Altland[3]
[1]*Fachbereich Physik, Universität Duisburg-Essen, 45117 Essen, Germany*
[2]*Institute of Physics, Saint-Petersburg University, 198504 Saint-Petersburg, Russia*
[3]*Institut für Theoretische Physik, Zülpicher Strasse 77, 50937 Köln, Germany*
(Received 15 January 2004; published 2 July 2004)

We sketch the semiclassical core of a proof of the so-called Bohigas-Giannoni-Schmit conjecture: A dynamical system with full classical chaos has a quantum energy spectrum with universal fluctuations on the scale of the mean level spacing. We show how in the semiclassical limit all system specific properties fade away, leaving only ergodicity, hyperbolicity, and combinatorics as agents determining the contributions of pairs of classical periodic orbits to the quantum spectral form factor. The small-time form factor is thus reproduced semiclassically. Bridges between classical orbits and (the nonlinear sigma model of) quantum field theory are built by revealing the contributing orbit pairs as topologically equivalent to Feynman diagrams.

DOI: 10.1103/PhysRevLett.93.014103 PACS numbers: 05.45.Mt, 03.65.Sq

Fully chaotic dynamics enjoy ergodicity and thus visit everywhere in the accessible space with uniform likelihood, over long periods of time. Even long periodic orbits bring about such uniform coverage. Moreover, classical ergodicity provides quantum chaos with universal characteristics.

Given chaos, quantum energy levels are correlated within local few-level clusters but become statistically independent as their distance grows much larger than the mean level spacing Δ. The decay of correlations on the scale Δ is empirically found system independent, within universality classes distinguished by the presence or the absence of time-reversal ($\mathcal{T}$) invariance [1,2]. Corresponding universal long-time characteristics act on the Heisenberg scale $T_H = 2\pi\hbar/\Delta$, with $\hbar$ Planck's constant.

Universal spectral fluctuations were conjectured as a manifestation of quantum chaos two decades ago [3]. Now, the semiclassical core of a proof can be given. Based on Gutzwiller's periodic-orbit theory [4], our progress comes with two surprises: one lies in its simplicity, the other in the appearance of interesting mathematics (nontrivial properties of permutations). Moreover, the often disputed intimate relation between periodic orbits and quantum field theory is confirmed for good. We thus expect the underlying ideas to radiate beyond spectral fluctuations, like to transport and localization.

Technically speaking, we want to show that each completely hyperbolic classical dynamics has a quantum energy spectrum with the same fluctuations as a random-matrix caricature $H_{\rm RMT}$ of its Hamiltonian, even though that caricature has nothing in common with the Hamiltonian but symmetry (absence or presence of $\mathcal{T}$ invariance). The theory of random matrices (RMT) [1,2,5], developed by Wigner and Dyson to account for fluctuations in nuclear spectra, yields analytic results for correlators of the level density $\rho(E)$, by averaging over suitable ensembles of random matrices. Simplest is the two-point correlator $\overline{\rho(E)\rho(E')} - \overline{\rho(E)}\,\overline{\rho(E')}$, where the overlines denote the ensemble average. Its Fourier transform with respect to the energy difference $E - E'$, called spectral form factor $K(\tau)$, is predicted by RMT for systems without $\mathcal{T}$ invariance (unitary class) and with that symmetry (orthogonal class) as

$$K_{\rm uni}(\tau) = \tau, \qquad K_{\rm orth}(\tau) = 2\tau - \tau\ln(1 + 2\tau), \qquad (1)$$

respectively; here τ is a time measured in units of the Heisenberg time T_H ranging in $0 \le \tau \le 1$. Note that the "orthogonal" form factor admits the expansion $K(\tau) = 2\tau - 2\tau^2 + 2\tau^3\cdots$ which converges for $0 \le \tau \le \frac{1}{2}$. Leaving larger times for future work we propose to show fidelity of individual chaotic dynamics to (1).

Of the many ways of doing RMT averages yielding (1), the quantum field-theoretical nonlinear sigma model [6,7] deserves special mention since it yields a τ expansion of the form factor equivalent to the semiclassical expansion to be developed here. The model points to analogies between hyperbolic dynamics and the motion of electrons in disordered media. In fact, the equivalence of semiclassical and field-theoretic expansions was first suggested in the context of disordered metals [8].

We start from Gutzwiller's representation of the level density of a hyperbolic system by a sum over its classical periodic orbits, $\rho(E) \propto {\rm Re}\sum_\gamma A_\gamma e^{iS_\gamma/\hbar}$, with $S_\gamma(E)$ the action and A_γ the (dimensionless) stability amplitude of the γth orbit. The form factor $K(\tau)$ is the double sum

$$K_{\rm po}(\tau) = \left\langle \sum_{\gamma,\gamma'} A_\gamma A_{\gamma'}^* e^{i(S_\gamma - S_{\gamma'})/\hbar}\,\delta\left(\tau - \frac{T_\gamma + T_{\gamma'}}{2T_H}\right)\right\rangle \qquad (2)$$

with $T_\gamma(E)$ the period of γ; the angular brackets demand averages over the energy and over a time interval $\Delta T \ll T_H$. We aim at evaluating the periodic-orbit sum (2) in the *semiclassical limit* $\hbar \to 0$, $T_\gamma \to \infty$ with $T_\gamma/T_H = $ const. That limit and the averages indicated eliminate noise due

 0031-9007/04/93(1)/014103(4)$22.50

to orbits with $|S_\gamma - S_{\gamma'}| \gg \hbar$ and purge $K(\tau)$ of system specific features.

For the formal double sum in (2) to converge to the RMT prediction (1), it must be structured into contributions from *families of orbit pairs*, such that each term of the τ expansion of $K(\tau)$ comes from a specific set of families. The simplest family contains the diagonal pairs $\{\gamma, \gamma\}$ and, given time-reversal invariance, $\{\gamma, \mathcal{T}\gamma\}$, where $\mathcal{T}\gamma$ is the time reverse of γ; it yields Berry's [9] "diagonal approximation" $K_{\rm po}^{(1)} = \kappa\tau$ where $\kappa = 1$ without and $\kappa = 2$ with $\mathcal{T}$ invariance, due to the doubling of pairs in the latter case. It is here, when summing over the "diagonal pairs" that we first meet the ergodicity of long periodic orbits, through Hannay–Ozorio de Almeida's (HOdA) [10] sum rule $\langle \sum_\gamma |A_\gamma|^2 \delta[\tau - (T_\gamma/T_H)]\rangle = \tau$. In a paradigmatic breakthrough, Sieber and Richter [11] gave the family responsible for the τ^2 term of $\mathcal{T}$ invariant dynamics; it is on the basis of their insight that we could find and account for all other families.

We first turn to the *unitary class* and propose to demonstrate that all families of orbit pairs individually contributing to higher orders τ^n collectively cancel for $n > 1$. To ease our task we assume two freedoms.

Long orbits have lots of close self-encounters. We speak of an l-encounter when l orbit stretches get and stay close for as long as their exponential divergence permits (Fig. 1). Since the closest approaches discernible quantum mechanically have an action scale $\hbar$ we expect relevant encounter durations $t_{\rm enc}$ of the order of the Ehrenfest time $T_E \sim \lambda^{-1}\ln({\rm const}/\hbar)$, with λ the Lyapunov rate of divergence. Departing from and ending on the $2l$ "ports" of an l-encounter are l "loops" with durations of the order of the period T and thus of the Heisenberg time $T_H = 2\pi\hbar/\Delta = \Omega/2\pi\hbar$, where Ω is the volume of the energy shell. Different encounters must be considered as separate: overlap of any two would yield a single one with more internal stretches. More generally, an orbit must leave an encounter before reentering it or another one.

Self-encounters lead us from an orbit γ to partners γ'. Two orbits in a pair $\{\gamma, \gamma'\}$ are practically indistinguishable in the loops outside encounters; they differ only within comparatively short encounters, by their connections of the outside loops. The action difference $S_\gamma - S_{\gamma'}$ can thus be of order $\hbar$. Reshuffling intraencounter connections of γ either yields a partner orbit γ' or a pseudo-orbit decomposing into shorter orbits; pseudo-orbits, Fig. 2, are not admitted to the Gutzwiller sum (2).

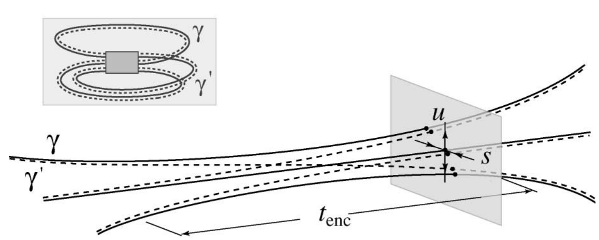

FIG. 1. A triple encounter ($l = 3$) in the energy shell and its Poincaré section for an orbit pair γ, γ'. Inset: Global appearance of the pair and the generating encounter.

Calling v_l the number of l-encounters ($l \geq 2$) within which γ and γ' differ by connections of the coinciding outside loops we write $V = \sum_l v_l$ for the total number of such encounters and $L = \sum_l l v_l$ for the number of orbit stretches within encounters (equalling the number of loops outside). We shall see that the families of orbit pairs with fixed $n = L - V + 1$ exclusively contribute to τ^n in $K(\tau)$. To calculate those contributions and check that they sum up to zero for $n > 1$ in the absence of $\mathcal{T}$ invariance, first we must understand the phase-space structure of self-encounters and, second, the combinatorics of counting proper partners must be mastered.

We begin with a closer analysis of self-encounters [12]. Drawing a Poincaré section, two dimensional for two-freedom systems, through an l-encounter we see the l orbit stretches of, say, γ pierce through that section in l points $x_i = (u_i, s_i)$, $i = 1 \cdots l$; one of these can be chosen as the origin of coordinate axes spanned by the unstable and stable manifolds of γ through $x_1 = (0, 0)$. For an encounter to be close we require $|u_i|, |s_i| \leq c$, with the bound c small enough for the motion along the l orbit stretches to allow for mutually linearized treatment. The $l - 1$ piercings $x_i \neq 0$ uniquely determine (i) the duration $t_{\rm enc}^{(l)}(u, s)$ of an l-encounter as a (logarithmic) function of the u_i, s_i [12], (ii) the piercings of the partner orbit(s), and (iii) the contribution to the action difference $S_\gamma - S_{\gamma'}$. There is a canonical transformation $u, s \to \tilde{u}, \tilde{s}$ diagonalizing the action difference to $\Delta S_{\rm enc}^{(l)} = \sum_{i=2}^{l} \tilde{u}_i \tilde{s}_i$. Both $t_{\rm enc}^{(l)}$ and $\Delta S_{\rm enc}^{(l)}$ are canonical invariants.

We characterize a set of encounters by a vector $\vec{v}$ whose components are the numbers v_l of l-encounters. We define a (weighted) number $w(u, s) d^{L-V}u d^{L-V}s$ of encounter sets with fixed vector $\vec{v}$ and temporal order of the L visits of the V encounters inside an orbit of period T; it contains a factor for each encounter involved, the fraction of its duration which the corresponding unstable and stable components spend in the intervals $[u_i, u_i + du_i]$ and $[s_i, s_i + ds_i]$. That number is determined by ergodicity as follows. A piercing of γ through a section will be found with the uniform probability $dt_i du ds/\Omega$ in a time interval $[t_i, t_i + dt_i]$ and in the area element $[u, u + du] \times [s, s + ds]$. We integrate the product of $L - V$ such probabilities over the L times t_i; here $t_1 \in [0, T]$ while we restrict the $L - 1$ other t_i to (i) a specific order in the interval $[t_1, t_1 + T]$ and (ii) by minimal separations (due to the ban of encounter overlap). To get the dimensionless

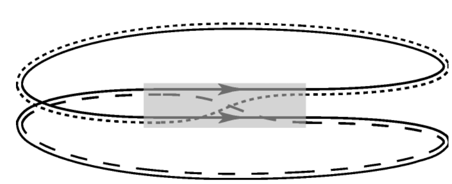

FIG. 2. Illustration of the counting problem: pseudo-orbits (here, dashed line) must be eradicated.

weight $w(u,s)d^{L-V}ud^{L-V}s$ we divide the L-fold integral by the product $\prod t_{\rm enc}^{(l)}$ of durations of the V encounters [12],

$$w(u,s)=\frac{T(T-\sum l t_{\rm enc}^{(l)})^{L-1}}{\Omega^{L-V}(L-1)!\prod t_{\rm enc}^{(l)}}; \qquad (3)$$

here, the restrictions mentioned in effect reduce the interval accessible to $L-1$ integration variables by the cumulative duration $t_{\rm excl}=\sum l t_{\rm enc}^{(l)}$ of the L intraencounter stretches. We note that the contribution $l t_{\rm enc}^{(l)}$ of each l-encounter to $t_{\rm excl}$ depends on the $l-1$ points of piercing $x=(u,s)\neq 0$ of γ through the pertinent section. We shall see below that the nonvanishing duration $t_{\rm excl}$ is, even though a small correction to the period T in $w(u,s)$, of decisive importance for spectral universality.

Remarkably, $w(u,s)$ results as independent of the order of the L passages of γ through the V self-encounters. We can therefore proceed to the number of self-encounters with fixed $\vec{v}$ and u, s irrespective of the order of visits, $N(\vec{v})w(u,s)$, by accounting for a multiplicity $N(\vec{v})$.

The number $N(\vec{v})$ brings up a combinatorial problem with a shade of topology mixed in (partner orbits must be connected), decoupled from the phase-space considerations yielding $w(u,s)$. When stating that γ and γ' differ in v_l l-encounters, $l=2,3,\ldots$, we leave open (i) the order in which the L encounter stretches are passed (in particular, which is the first) and (ii) how the intraencounter connections of γ are reshuffled in γ'. The number of possibilities left is $N(\vec{v})$; we can determine it by running through all different orderings of visits as well as through all intraencounter connections other than the one realized by γ and checking, in each case, whether an orbit or a pseudo-orbit results. For only a few v_l nonzero, $N(\vec{v})$ is easily found with paper, pencil, and patience. For general $\vec{v}$, the permutation problem at issue can be attacked recursively as discussed in the technical note below.

With the help of the weight $N(\vec{v})w(u,s)$ we replace the sum over orbit pairs in (2) as $\sum_{\gamma\gamma'}\to\sum_\gamma\sum_{\vec{v}}N(\vec{v})\times\int d^{L-V}ud^{L-V}s\,w(u,s)L^{-1}$. The number of loops L had to be divided out here, since the L choices of one intraencounter stretch as the first yield the same partner orbit. The summand simplifies as $A_\gamma A^*_{\gamma'}\to|A_\gamma|^2$, $T_\gamma+T_{\gamma'}\to 2T_\gamma$ since in contrast to the action difference ΔS, the prefactors and periods suffer no relative discrimination by a small quantum unit. We may also invoke the HOdA sum rule already met above, to do the sum over γ, and thus find the form factor as

$$\frac{K(\tau)-\tau}{\tau}=\sum_{\vec{v}}N(\vec{v})\int_{-c}^{c}d^{L-V}ud^{L-V}s\,\frac{w(u,s)e^{i\Delta S/\hbar}}{L}. \qquad (4)$$

Only a single term of the multinomial expansion of $(T-t_{\rm excl})^{L-1}$ in $w(u,s)$ survives the limit $\hbar\to 0$, the one which cancels the denominator $\prod t_{\rm enc}^{(l)}$ and comes with the factor $(T/\Omega)^{L-V}$; all other terms vanish, either because they involve too low orders in the period T and thus extra factors $\hbar$ besides a power of T/T_H or because they oscillate rapidly and are annulled by averaging over a small time window. The remaining integral simplifies after the canonical transformation diagonalizing ΔS, $\int_{-c}^{c}d^{L-V}ud^{L-V}s e^{i\Delta S/\hbar}=(\int_{-c}^{c}duds e^{ius/\hbar})^{L-V}\to(2\pi\hbar)^{L-V}$. With $(T2\pi\hbar/\Omega)^{L-V}=\tau^{L-V}$ we get the series $K(\tau)=\tau+\sum_{n=2}^{\infty}K_n\tau^n$ with the coefficient

$$K_n=\frac{1}{(n-2)!}\sum_{\vec{v}}^{(n=L-V+1)}N(\vec{v})\frac{(-1)^V\prod_l l^{v_l}}{L} \qquad (5)$$

governed by ergodicity, combinatorics, and topology, given the sets of separated close self-encounters.

The vanishing of the foregoing sum over families of partner orbits is a property of the permutation group which to the best of our knowledge was never noticed before. We sketch the surprisingly simple proof of $K_n=0$, for $n>1$ based on a recursion scheme for $N(\vec{v})$, in the technical note below. Universal spectral fluctuations are thus established for dynamics without $\mathcal{T}$ invariance.

The *orthogonal class* of $\mathcal{T}$ invariant dynamics can be treated similarly. We must generalize the notion of self-encounters to include orbit stretches close to the others *up to time reversal.* Configuration-space pictures prove useful: an orbit stretch may be depicted by an arrow $\rightarrow$. While all l-encounters admitted in the unitary case involve l parallel such arrows (like $\rightrightarrows$ or $\overrightarrow{\rightrightarrows}$), we now face, in addition, arrows with opposite directions (like $\rightleftarrows$ or $\overrightarrow{\leftrightarrows}$). Likewise, loops in between self-encounters of an orbit γ appear nearly unchanged in partner(s) γ', except that the senses of traversal may be opposite.

The multiplicity $N(\vec{v})$ of encounter "classes" $\vec{v}$ leads to a permutation problem slightly more complicated than in the unitary case. Again, all classes with fixed $n=L-V+1$ contribute to τ^n. The results (4) and (5) reappear with an additional factor of 2 due to the fact that with γ' a partner so is $\mathcal{T}\gamma'$. As discussed in the technical note, a recursion relation arises for K_n which yields the random-matrix form factor for the orthogonal universality class, $K_{\rm orth}=2\tau+\sum_{n\geq 2}[(-2)^{n-1}/(n-1)]\tau^n$. We would like to underscore that in establishing both the unitary and the orthogonal form factors as universal we have accounted for *all* orbit pairs whose members differ by nothing but the way almost coinciding (up to time reversal) loops are connected within close self-encounters.

The τ expansion of $K_{\rm orth}$ converges for $0\leq\tau\leq\frac{1}{2}$. The summed up logarithm remains valid, by analytic continuation, up to the next singularity. Neither the locus of that singularity ($\tau=1$) nor the form factor for $\tau>1$ can be found within the τ expansion. We underscore once more ergodicity and hyperbolicity as our basic assumptions; in addition, strong action degeneracies as for dynamics with Hecke symmetries must be excluded [13].

We must discuss the relation of our semiclassical work to the zero dimensional sigma model of quantum field theory [2,6,7]. The relevance of the sigma model for us lies in similarities of its perturbative implementation to our semiclassical expansion. A perturbative evaluation of the sigma model involves Wick's theorem which can be shown to entail a recursive reduction scheme equivalent to the topological and combinatorial problem yielding our multiplicity $N(\vec{v})$. Moreover, our orbit pairs correspond to the Feynman diagrams depicting terms of the perturbative treatment of the sigma model, with our l-encounters and the outside loops the analogs of vertices (with $2l$ ports) and propagator lines, respectively. Order by order in τ, our families of orbit pairs are equivalent to the Feynman diagrams of the sigma model [8].

Technical note.—We want to set up the permutation problem yielding the mutiplicity $N(\vec{v})$, first for the *unitary case*. To that end, starting from an arbitrary orbit stretch in some encounter we number the L stretches in the order of visits by γ. More precisely, we denote entrance ports of encounters by $1, 2, \ldots, L$ and exit ports by $1', 2', \ldots, L'$, such that the kth stretch of γ connects ports k and k'. In a partner orbit γ' of γ port k is connected to a port $j_k' \neq k'$. The L intraencounter stretches of γ thus correspond to the trivial permutation $P_E^\gamma = \binom{1,\ 2,\ \ldots, L}{1'\ 2', \ldots, L'}$ while a partner γ' will have intraencounter connections according to $P_E = \binom{1, \ldots, L}{j_1', \ldots, j_L'} \neq P_E^\gamma$. Since reconnections take place only within encounters, P_E must be composed of v_l cycles of length l, $l = 2, 3, \ldots$. (Mathematically speaking, P_E must belong to the conjugacy class $2^{v_2} 3^{v_3} \cdots l^{v_l}$ of the group of permutations of L objects corresponding to the cycles defined by the vector $\vec{v}$.)

The loops common to γ and its partners γ' are associated with the permutation $P_L = \binom{1'\ 2', \ldots, L'}{2\ \ 3, \ldots, 1}$. The whole of γ is represented by the product $P_L P_E^\gamma = \binom{1\ 2, \ldots, L}{2\ 3, \ldots, 1}$ and that product is a single-cycle permutation since γ is a single orbit, rather than a decomposing pseudo-orbit. Moreover, the product $P_L P_E$ describes a connected partner γ' rather than a decomposing pseudo-orbit if and only if it is single-cycle as well. The multiplicity $N(\vec{v})$ is thus found by running P_E through all possibilities and counting only those for which $P_L P_E$ is single-cycle.

In the *orthogonal case* there are 2^{l-1} distinct orientations of the stretches of an l-encounter. After combining oriented loops with reconnections of encounter stretches we must again check connectivity.

We have established a general recursion relation for the multiplicity $N(\vec{v})$, both in the unitary and the orthogonal cases, by following the change of $N(\vec{v})$ as (i) two encounters unite ($v_l \to v_l - 1$, $v_{l'} \to v_{l'} - 1$, $v_{l+l'-1} \to v_{l+l'-1} + 1$, in short $\vec{v} \to \vec{v}^{[l,l']}$), (ii) as an encounter splits into two, and (iii) as an l-encounter becomes an $(l-1)$-encounter by uniting two of its orbit stretches. The relations are best written for $\tilde{N}(\vec{v}) \equiv N(\vec{v})(-1)^V \times \prod_l l^{v_l} [L(n-2)!]^{-1}$; note that $K_n = \kappa \sum_{\vec{v}}^{(n=L-V+1)} \tilde{N}(\vec{v})$.

In the unitary case we only need the special variant of the general recursion relation concerning 2-encounters merging with l-encounters, where the recursion reads $v_2 \tilde{N}(\vec{v}) + \sum_{l \geq 2} l(v_{l+1} + 1) \tilde{N}(\vec{v}^{[l,2]}) = 0$. Summing over $\vec{v}$ with fixed $n = L - V + 1$ we obtain $\sum_{\vec{v}}^{(n=L-V+1)} \times [v_2 \tilde{N}(\vec{v}) + \sum_{l \geq 2} l(v_{l+1} + 1) \tilde{N}(\vec{v}^{[l,2]})] = 0$. In the foregoing double sum we can replace $\vec{v}^{[l,2]} \to \vec{v}$ and thus $v_{l+1} + 1 \to v_{l+1}$. The sum now runs over all $\vec{v}$ with $v_{l+1} > 0$; however, the latter restriction is immaterial due to the factor v_{l+1} in the summand. We thus obtain $\sum_{\vec{v}}^{(n=L-V+1)} [v_2 + \sum_{l \geq 2} l v_{l+1}] \tilde{N}(\vec{v}) = 0$; here, the term in the square bracket equals $n - 1$. The resulting identity $(n-1) K_n = 0$ implies $K_n = 0$ for $n > 1$.

In the orthogonal case we need two special cases of the general recursion relation, to account for the "disappearance" of 2-encounters *and* 3-encounters. A suitable linear combination yields $K_{n+1} = -[2(n-1)/n] K_n$; the latter recursion gives the random-matrix form factor.

Financial support by the SFB/TR12 of the Deutsche Forschungsgemeinschaft is gratefully acknowledged. We have enjoyed discussions with Jürgen Müller, Dominique Spehner, and Martin Zirnbauer.

[1] H.-J. Stöckmann, *Quantum Chaos: An Introduction* (Cambridge University Press, Cambridge, England, 1999).

[2] F. Haake, *Quantum Signatures of Chaos* (Springer, Berlin, 2000).

[3] O. Bohigas, M. J. Giannoni, and C. Schmit, Phys. Rev. Lett. **52**, 1 (1984); G. Casati, F. Valz-Gris, and I. Guarneri, Lett. Nuovo Cimento Soc. Ital. Fis. **28**, 279 (1980); M. V. Berry, Ann. Phys. (N.Y.) **131**, 163 (1981).

[4] M. Gutzwiller, *Chaos in Classical and Quantum Mechanics* (Springer, New York, 1990).

[5] M. L. Mehta, *Random Matrices and the Statistical Theory of Spectra* (Academic, New York, 1991), 2nd ed.

[6] F. Wegner, Z. Phys. B **35**, 207 (1979)

[7] K. Efetov, *Supersymmetry in Disorder and Chaos* (Cambridge University Press, Cambridge, England, 1997).

[8] R. A. Smith, I. V. Lerner, and B. L. Altshuler, Phys. Rev. B **58**, 10 343 (1998); R. S. Whitney, I. V. Lerner, and R. A. Smith, Waves Random Media **9**, 179 (1999).

[9] M. V. Berry, Proc. R. Soc. London, Ser. A **400**, 229 (1985).

[10] J. H. Hannay and A. M. Ozorio de Almeida, J. Phys. A **17**, 3429 (1984).

[11] M. Sieber and K. Richter, Phys. Scr. **T90**, 128 (2001); M. Sieber, J. Phys. A **35**, L613 (2002).

[12] See S. Heusler, S. Müller, P. Braun, and F. Haake, J. Phys. A **37**, L31 (2004) and references therein for 2-encounters and 3-encounters.

[13] E. Bogomolny and C. Schmit, J. Phys. A **37**, 4501 (2004), and references therein.