Mass of the η_b and α_s from the Nonrelativistic Renormalization Group

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We sum up the next-to-leading logarithmic corrections to the heavy-quarkonium hyperfine splitting, using the nonrelativistic renormalization group. On the basis of this result, we predict the mass of the η_b meson to be $M(\eta_b) = 9421 \pm 11(\mathrm{th})^{+9}_{-8}(\delta\alpha_s)$ MeV. The experimental measurement of $M(\eta_b)$ with a few MeV error would be sufficient to determine $\alpha_s(M_Z)$ with an accuracy of ± 0.003. For the hyperfine splitting in charmonium, the use of the nonrelativistic renormalization group brings the perturbative prediction significantly closer to the experimental figure.

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The theoretical study of nonrelativistic heavy-quark-antiquark systems is among the earliest applications of perturbative quantum chromodynamics (QCD) [1] and has by now become a classical problem. Its applications to bottomonium or toponium physics entirely rely on the first principles of QCD. This makes heavy-quark-antiquark systems an ideal laboratory to determine fundamental parameters of QCD, such as the strong-coupling constant α_s and the heavy-quark masses m_q. Besides its phenomenological importance, the heavy-quarkonium system is also very interesting from the theoretical point of view because it possesses highly sophisticated multiscale dynamics, and its study demands the full power of the effective-field-theory approach. The properties of the Υ mesons, the bottom quark-antiquark spin-one bound states, are measured experimentally with great precision, and recent theoretical analysis of the Υ family based on high-order perturbative calculations resulted in determinations of the bottom-quark mass m_b with unprecedented accuracy [2–4]. In contrast to the Υ family, the current experimental situation with the spin-zero η_b meson is rather uncertain [5]. Yet, the discovery of the η_b meson is one of the primary goals of the CLEO-c research program [6]. An accurate prediction of its mass $M(\eta_b)$ is thus a big challenge and a test for the QCD theory of heavy quarkonium. Moreover, the hyperfine splitting (HFS) of the bottomonium ground state, $E_{\mathrm{HFS}} = M(\Upsilon(1S)) - M(\eta_b)$, is very sensitive to α_s and, with the advancement of the experimental measurements, could become a competitive source for the determination of α_s.

The HFS in quarkonium has been a subject of several theoretical researches [7]. To our knowledge, the next-to-leading order (NLO) $\mathcal{O}(\alpha_s)$ correction is currently known in a closed analytical form only for the ground state HFS [4]. In this Letter, we generalize this result to the excited states and present the analytical renormalization-group-improved expression for the heavy-quarkonium HFS in the next-to-leading logarithmic (NLL) approximation, which sums up all the corrections of the form $\alpha_s^n \ln^{n-1}\alpha_s$. We apply it to predict $M(\eta_b)$, which could be used for extracting α_s from future experimental data on the η_b.

The leading-order (LO) result for the HFS is proportional to the fourth power of α_s, $E_{\mathrm{HFS}}^{\mathrm{LO}} = C_F^4 \alpha_s^4(\mu) m_q/(3n^3)$, where $C_F = (N_c^2 - 1)/(2N_c)$, and suffers from a strong dependence on the renormalization scale μ of $\alpha_s(\mu)$, which essentially limits the numerical accuracy of the approximation. Thus, the proper fixing of μ is mandatory for the HFS phenomenology. The scale dependence of a finite-order result is canceled against the higher-order logarithmic contributions proportional to a power of $\ln(\mu/\bar{\mu})$, where $\bar{\mu}$ is a dynamical scale of the nonrelativistic bound-state problem. The *physical* choice of the scale $\mu = \bar{\mu}$ eliminates these potentially large logarithmic terms and *a priori* minimizes the scale dependence. However, the dynamics of the nonrelativistic bound state is characterized by three well-separated scales: the hard scale of the heavy-quark mass m_q, the soft scale of the bound-state momentum $v m_q$, and the ultrasoft scale of the bound-state energy $v^2 m_q$, where $v \propto \alpha_s$ is the velocity of the heavy quark inside the approximately Coulombic bound state. To make the procedure of scale fixing self-consistent, one has to resum to all orders the large logarithms of the scale ratios characteristic for the nonrelativistic bound-state problem. The resummation of the logarithmic corrections requires an appropriate conceptual framework. The effective field theory [8] is now recognized as a powerful tool for the analysis of multiscale systems, which is at the heart of the recent progress in the perturbative QCD bound-state calculations. The main idea of this method is to decompose the complicated multiscale problem into a sequence of simpler problems, each involving a smaller number of scales. The logarithmic corrections originate from logarithmic integrals over virtual momenta ranging between the scales and reveal themselves as the singularities of the effective-theory couplings. The renormalization of these singularities allows one to derive the equations of the *nonrelativistic renormalization group* (NRG), which describe the running of the effective-theory couplings, i.e., their dependence on the effective-theory cutoffs. The solution of these equations sums up the logarithms of the scale ratios. To derive the NRG equations necessary for the NLL analysis of the HFS, we rely on the method based on the formulation of the nonrelativistic effective theory known as potential nonrelativistic QCD (pNRQCD) [9]. The method was developed in Ref. [10] where, in particular, the leading logarithmic (LL) result for the HFS has been obtained (see also Ref. [11]). A characteristic feature of the NRG is the correlation of the dynamical scales, which leads to the correlation of the cutoffs [12]. For perturbative calculations within the effective theory, dimensional regularization is used to handle the divergences, and the formal expressions derived from the Feynman rules of the effective theory are understood in the sense of the threshold expansion [13]. This approach [14–17] possesses two crucial virtues: the absence of additional regulator scales and the automatic matching of the contributions from different scales.

Let us give a few details of the NLL analysis. We distinguish the soft, potential, and ultrasoft anomalous dimensions corresponding to the ultraviolet divergences of the soft, potential, and ultrasoft *regions* [13]. The LL approximation is determined by the one-loop soft running of the effective Fermi coupling c_F and the spin-flip four-quark operator [10]. In the NLL approximation, all three types of running contribute. We need the two-loop soft running of c_F, which is known [18], and the two-loop soft running of the spin-flip four-quark operator, which we compute by adopting the technique used in Ref. [14] for the calculation of the two-loop $1/(m_q r^2)$ non-Abelian potential. To compute the potential running, we inspect all operators that lead to spin-dependent ultraviolet divergences in the time-independent perturbation theory contribution with one and two potential loops [10,19]. They include (i) the $\mathcal{O}(v^2, \alpha_s v)$ operators [2,7], (ii) the tree $\mathcal{O}(v^4)$ operators, some of which can be checked against the QED analysis [16], and (iii) the one-loop $\mathcal{O}(\alpha_s v^3)$ operators, for which only the Abelian parts are known [16], while the non-Abelian parts are new. In the NLL approximation, we need the LL soft and ultrasoft running of the $\mathcal{O}(v^2)$ and $\mathcal{O}(v^4)$ operators, which enter the two-loop time-independent perturbation theory diagrams, and the NLL soft and ultrasoft running of the $\mathcal{O}(\alpha_s v)$ and $\mathcal{O}(\alpha_s v^3)$ operators, which contribute at one-loop. The running of the $\mathcal{O}(v^2, \alpha_s v)$ operators is already known within pNRQCD [10]. The running of the other operators is new. For some of them, it can be obtained using reparametrization invariance [20].

Besides the running discussed above, we need the initial conditions for the NRG evolution given by the known one-loop result [7]. With the anomalous dimensions and initial conditions at hand, it is straightforward to solve the system of the nonlinear differential equations for the effective couplings and get the NLL result for the HFS. The corresponding expression for general color (light-flavor) number N_c (n_l) and for arbitrary principal quantum number n is too lengthy to be shown in this Letter, so we present the explicit analytical expression only for $N_c = 3$, $n_l = 4$, and $n = 1$, which applies to the bottomonium ground state. It reads

$$
\begin{aligned}
E_{\rm HFS}^{\rm NLL} = \frac{C_F^4 \alpha_s^4(\mu) m_b}{3}\Bigg\{ & \frac{27}{14} y^{-1} - \frac{13}{14} y^{-18/25} + \frac{\alpha_s(m_b)}{\pi}\Bigg[\left(\frac{1037}{224} + \frac{405086361761\pi^2}{25617160800} - \frac{3}{4}\ln 2\right) y^{-1} - \frac{1024\pi^2}{143} y^{-39/50} \\
& - \left(\frac{102973}{26250} + \frac{184336\pi^2}{25725}\right) y^{-18/25} + \frac{1024\pi^2}{675} y^{-1/2} + \frac{671\pi^2}{1029} y^{-11/25} - \frac{3\pi^2}{23} y^{-2/25} + \left(-\frac{13427921}{1260000} + \frac{88057\pi^2}{151200}\right) y^{7/25} \\
& + \frac{4\pi^2}{41} y^{16/25} + \frac{1377}{56} - \frac{1253587\pi^2}{227500} - \frac{629\pi^2}{7500} y - \frac{2873\pi^2}{7182} y^{32/25}\, {}_2F_1\!\left(\frac{57}{25}, 1; \frac{82}{25}; \frac{y}{2}\right) + \frac{2873\pi^2}{3591} y^{-1}\, {}_2F_1\!\left(1, 1; \frac{82}{25}; -1\right) \\
& + \left(\frac{675}{28} - \frac{533}{42} y^{7/25}\right)\ln\left(\frac{\mu}{\bar\mu}\right) + \frac{85248\pi^2}{30625} y^{-1}\ln y + \left(-\frac{45834}{4375} y^{-1} + \frac{21216}{4375} - \frac{2873}{1575} y^{7/25} + \frac{243}{1250} y\right)\pi^2 \ln(2-y)\Bigg]\Bigg\}, \qquad (1)
\end{aligned}
$$

where α_s is renormalized in the $\overline{\rm MS}$ scheme, $y = \alpha_s(\mu)/\alpha_s(m_b)$, $\bar\mu = C_F \alpha_s(\mu) m_b$, ${}_2F_1(a, b; c; z)$ is the hypergeometric function, and ${}_2F_1(1, 1; 82/25; -1) = 0.7875078\ldots$. By expanding the resummed expression up to $\mathcal{O}(\alpha_s^2)$, we get

$$
\begin{aligned}
E_{\rm HFS}^{\rm NLL} = E_{\rm HFS}^{\rm LO}\Bigg\{ & 1 + \frac{\alpha_s}{\pi}\Bigg[\left(\frac{1}{2} + \frac{5}{4n} - n\Psi_2(n)\right)\beta_0 + \left(-\frac{37}{72} - \frac{7}{8n} + \frac{7}{4}[\Psi_1(n+1) + \gamma_E + L_{\alpha_s}^n]\right) C_A - \frac{C_F}{2} + \frac{3}{2}(1 - \ln 2) T_F \\
& - \frac{5}{9} n_l T_F\Bigg] + \left(\frac{\alpha_s}{\pi}\right)^2 L_{\alpha_s}^n \Bigg[\left(\frac{19}{6} C_A^2 - \frac{5}{6} C_A n_l T_F\right) L_{\alpha_s}^n + \frac{\beta_1}{8} \\
& + \left(\left(\frac{169}{144} + \frac{2}{n} + \frac{3}{8}[\Psi_1(n+1) + \gamma_E] - \frac{7}{4} n\Psi_2(n)\right) C_A - C_F + \frac{3}{2}(1 - \ln 2) T_F\right)\beta_0 \\
& + \left(-\frac{23}{27} - \frac{\pi^2}{6} - \frac{7}{8n} + \frac{7}{4}[\Psi_1(n+1) + \gamma_E]\right) C_A^2 + \left(\frac{7}{4} - \frac{11\pi^2}{8}\right) C_A C_F - \frac{\pi^2}{2} C_F^2 - \frac{44}{27} C_A T_F n_l + \frac{1}{2} C_F T_F n_l\Bigg]\Bigg\}, \qquad (2)
\end{aligned}
$$

where $\alpha_s \equiv \alpha_s(\mu)$, $\mu = \bar{\mu}/n$, $C_A = N_c$, $T_F = 1/2$, β_i is the $(i+1)$-loop coefficient of the QCD β function ($\beta_0 = 11C_A/3 - 4T_F n_l/3, \ldots$), $L^n_{\alpha_s} = \ln(C_F\alpha_s/n)$, $\Psi_n(x) = d^n \ln\Gamma(x)/dx^n$, $\Gamma(x)$ is Euler's Γ function, and $\gamma_E = 0.577\,216\ldots$ is Euler's constant. In Eq. (2), we keep the full dependence on N_c, n_l, and n. The $\mathcal{O}(\alpha_s^2\ln^2\alpha_s)$ term is known [10,11], while the $\mathcal{O}(\alpha_s^2\ln\alpha_s)$ term is new.

For the numerical estimates, we adopt the following strategy. We take $m_b = M(\Upsilon(1S))/2$, which is sufficient at the order of interest. Furthermore, we take $\alpha_s(M_Z)$ as an input and run with four-loop accuracy down to the matching scale m_b to ensure the best precision. Below the matching scale, the running of α_s is used according to the logarithmic precision of the calculation in order not to include next-to-next-to-leading logarithms in our analysis. In Fig. 1, the HFS for the bottomonium ground state is plotted as a function of μ in the LO, NLO, LL, and NLL approximations. As we see, the LL curve shows a weaker scale dependence compared to the LO one. The scale dependence of the NLO and NLL expressions is further reduced, and, moreover, the NLL approximation remains stable up to smaller scales than the fixed-order calculation. At the scale $\mu' \approx 1.3$ GeV, which is close to the inverse Bohr radius, the NLL correction vanishes. Furthermore, at $\mu'' \approx 1.5$ GeV, where $\alpha_s^{\rm LL} = 0.319$, the result becomes independent of μ; i.e., the NLL curve shows a local maximum. This suggests a nice convergence of the logarithmic expansion despite the presence of the ultrasoft contribution with α_s normalized at the rather low scale $\bar{\mu}^2/m_b \sim 0.8$ GeV. By taking the difference of the NLL and LL results at the local maxima as a conservative estimate of the error due to uncalculated higher-order contributions, we get $E_{\rm HFS} = 39 \pm 8$ MeV. A similar error estimate is obtained by the variation of the normalization scale in the physically motivated soft region 1–3 GeV.

So far, we have only discussed the perturbative contributions to the HFS. The nonperturbative ones are in general given by the convolution of a quantum-mechanical Green function with a nonlocal nonperturbative gluonic correlator [21]. In the limit $\alpha_s^2 m_q \ll \Lambda_{\rm QCD}$, it can be investigated by the method of vacuum condensate expansion [22]. The resulting series, however, is not expected to converge well in our case and suffers from large numerical uncertainties [23]. In any case, within the power counting assumed in this Letter, these nonperturbative effects are beyond the accuracy of our computation and should be added to the errors. One way to estimate them is by considering the HFS in the charmonium system, where experimental data are available. The result of our analysis is given in Fig. 2 along with the experimental value 117.7 ± 1.3 MeV [24]. The local maximum of the NLL curve corresponds to $E_{\rm HFS} = 104$ MeV and $\alpha_s^{\rm LL} = 0.534$. We should emphasize the crucial role of the resummation to bring the perturbative prediction closer to the experimental figure. Note also that the recent lattice estimates undershoot the experimental value by 20–30% [25]. For an estimate, we attribute the whole difference of ≈ 14 MeV to the nonperturbative effects. Taking into account that they are suppressed by the inverse heavy-quark mass at least as $1/(\alpha_s m_q)^2$ [22], we obtain ≈ 3.5 MeV for the typical size of the nonperturbative contribution to the HFS in bottomonium. For the estimate of the nonperturbative error, we multiply this number by two.

Our prediction for the bottomonium HFS can be compared with those obtained either on the lattice [26] or with potential models (for a recent discussion see Ref. [27], and references therein). It seems to be a general trend that our result is larger than the lattice predictions and smaller than most of the potential model results.

To conclude, we have computed the heavy-quarkonium HFS in the NLL approximation by summing up the subleading logarithms of α_s to all orders in the perturbative expansion. The use of the NRG extends the range of μ

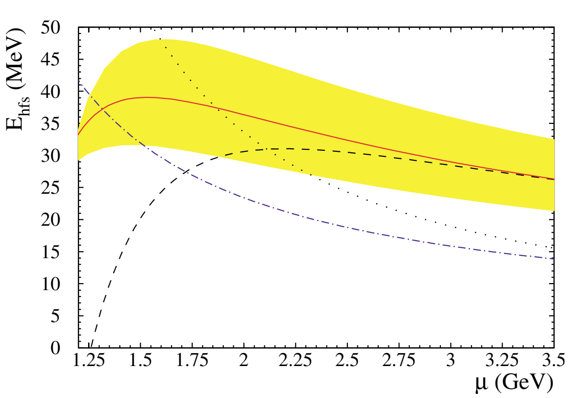

FIG. 1 (color online). HFS of $1S$ bottomonium as a function of the renormalization scale μ in the LO (dotted line), NLO (dashed line), LL (dot-dashed line), and NLL (solid line) approximations. For the NLL result, the band reflects the errors due to $\alpha_s(M_Z) = 0.118 \pm 0.003$.

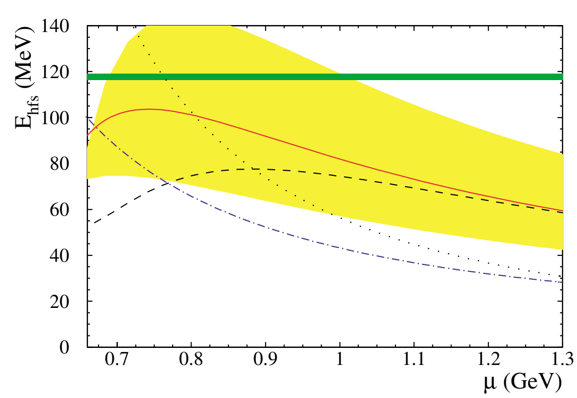

FIG. 2 (color online). HFS of $1S$ charmonium as a function of the renormalization scale μ in the LO (dotted line), NLO (dashed line), LL (dot-dashed line), and NLL (solid line) approximations. For the NLL result, the band reflects the errors due to $\alpha_s(M_Z) = 0.118 \pm 0.003$. The horizontal band gives the experimental value 117.7 ± 1.3 MeV [24].

where the perturbative result is stable to the physical scale of the inverse Bohr radius. We found the resummation of logarithms to be crucial to bring the perturbative prediction closer to the experimental figure of the HFS in charmonium despite *a priori* unsuppressed nonperturbative effects. Our results further indicate that the properties of the physical charmonium and bottomonium ground states are dictated by perturbative dynamics. As an application of the result to the bottomonium spectrum, we predict the mass of the as yet undiscovered η_b meson to be

$$M(\eta_b) = 9421 \pm 11(\mathrm{th})^{+9}_{-8}(\delta\alpha_s)\ \mathrm{MeV}, \tag{3}$$

where the errors due to the high-order perturbative corrections and the nonperturbative effects are added up in quadrature in "th," whereas "$\delta\alpha_s$" stands for the uncertainty in $\alpha_s(M_Z) = 0.118 \pm 0.003$. If the experimental error in future measurements of $M(\eta_b)$ will not exceed a few MeV, the bottomonium HFS will become a competitive source of $\alpha_s(M_Z)$ with an estimated accuracy of ± 0.003, as can be seen in Fig. 1.

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