Complexity of Random Energy Landscapes, Glass Transition, and Absolute Value of the Spectral Determinant of Random Matrices

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Finding the mean of the total number N_{tot} of stationary points for N-dimensional random energy landscapes is reduced to averaging the absolute value of the characteristic polynomial of the corresponding Hessian. For any finite N we provide the exact solution to the problem for a class of landscapes corresponding to the "toy model" of manifolds in a random environment. For $N \gg 1$ our asymptotic analysis reveals a phase transition at some critical value μ_c of a control parameter μ from a phase with a finite landscape complexity: $N_{\mathrm{tot}} \sim e^{N\Sigma}$, $\Sigma(\mu < \mu_c) > 0$ to the phase with vanishing complexity: $\Sigma(\mu > \mu_c) = 0$. Finally, we discuss a method of dealing with the modulus of the spectral determinant applicable to a broad class of problems.

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Characterizing geometry of a complicated landscape described by a random function $\mathcal{H}$ of N real variables $\mathbf{x} = (x_1, ..., x_N)$ is an important problem motivated by numerous applications in physics, image processing, and other fields of applied mathematics [1]. The simplest, yet nontrivial task [2–6], is to find the mean number of all stationary points of $\mathcal{H}$ (minima, maxima, and saddles) in a given domain of the Euclidean space by investigating the simultaneous stationarity conditions $\partial_k \mathcal{H} = 0$ for all $k = 1, ..., N$, with ∂_k standing for the partial derivative $\partial/\partial x_k$. In this context the function

$$\mathcal{H} = \frac{\mu}{2}\sum_{k=1}^{N} x_k^2 + V(x_1, \ldots, x_N), \qquad (1)$$

given by the sum of a purely deterministic quadratic piece characterized by a non-negative parameter $\mu \geq 0$ and of a random Gaussian function $V(\mathbf{x})$, attracted considerable interest for several independent reasons. For small $N = 1, 2$ statistics of stationary points of (1) were investigated long ago in a classical study of specular light reflection from a random sea surface [2] and addressed several times since in various physical contexts; see [3,4]. Most frequently one assumes the Gaussian part to be isotropic with zero mean and correlations depending only on the Euclidean distance $|\mathbf{x}_1 - \mathbf{x}_2|^2 = \sum_{k=1}^{N}(x_{1k} - x_{2k})^2$ and given by

$$\langle V(\mathbf{x}_1)V(\mathbf{x}_2)\rangle = N f\left[\frac{1}{2N}|\mathbf{x}_1 - \mathbf{x}_2|^2\right], \qquad (2)$$

with the brackets standing for the ensemble average.

Recently much interest in Eq. (1) was boosted by reinterpreting it as the energy functional of a certain "toy" model describing elastic manifolds propagating in a random potential; see [7,8], and references therein. This type of model is known to display a very nontrivial glassy behavior at low enough temperatures—an unusual off-equilibrium relaxation dynamics attributed to a complex structure of their energy landscape. Although particular dynamical as well as statical properties may differ substantially for different functions $f(x)$ (e.g., "long-range" vs "short-range" correlated potentials; see [8]), the very fact of glassy relaxation is common to all of them. In fact, the same model admits an alternative interpretation as a spin glass, with x_i being looked at as "soft spins" in a quadratic well interacting via a random potential V [7]. From this point of view it is most interesting to concentrate on the limit of a large number of "spins," $N \gg 1$. The experience accumulated from working with various types of spin-glass models [5] suggests that, for the energy landscape to be complex enough to induce a glassy behavior the total number of stationary points $N_{\mathrm{tot}}(\mu)$ should grow exponentially with N as $N_{\mathrm{tot}}(\mu) \sim \exp N\Sigma(\mu)$. The quantity $\Sigma(\mu) > 0$ in such a context is natural to call the *landscape complexity.* On the other hand, it is completely clear that the number of stationary points should tend to $N_{\mathrm{tot}} = 1$ for very large μ when the random part is negligible in comparison with the deterministic one. In fact, when $N \to \infty$ we will find that a kind of sharp transition to the phase with vanishing complexity occurs at some finite critical value μ_c, so that $\Sigma(\mu) = 0$ as long as $\mu > \mu_c$, whereas $\Sigma(\mu) > 0$ for $\mu < \mu_c$ and tends to zero quadratically when $\mu \to \mu_c$. Such a transition is just the glass transition observed earlier in a framework of a different approach in [7,8].

We start with writing the number of stationary points of $\mathcal{H}$ in any spatial domain D as $N_{\mathrm{tot}}^{(D)} = \int_D \rho(\mathbf{x})\, d^N\mathbf{x}$, with $\rho(\mathbf{x})$ being the corresponding density of the stationary points. The ensemble-averaged value of such a density can be found as

$$\rho_{\mathrm{av}}(\mathbf{x}) = \left\langle |\det(\mu\delta_{k_1,k_2} + \partial^2_{k_1,k_2}V)| \prod_{k=1}^{N} \delta(\mu x_k + \partial_k V)\right\rangle,$$

where $\delta(x)$ and δ_{mn} stand for the Dirac's δ function and Kroneker symbol, respectively.

To evaluate the ensemble average we notice that for the Gaussian potential V the first derivatives $\partial_k V$ are Gaussian distributed and are locally statistically independent of the second derivatives. Representing the δ functions as Fourier integrals and exploiting $\langle \partial_m V \partial_n V \rangle = a^2 \delta_{mn}$, $a^2 = -f'(0)$ one can easily perform the corresponding part of the averaging and arrive at

$$\rho_{\mathrm{av}}(\mathbf{x}) = \frac{1}{[\sqrt{2\pi}a]^N} e^{-(\mu^2 \mathbf{x}^2/2a^2)} \langle |\det(\mu \delta_{k_1,k_2} + H_{k_1 k_2})| \rangle, \tag{3}$$

where we introduced the matrix of second derivatives of the potential $H_{k_1 k_2} \equiv \partial^2_{k_1,k_2} V$. Further changing $H \to -H$ we see that the problem basically amounts to evaluating the ensemble average of the absolute value of the characteristic polynomial $\det(\mu I_N - H)$ (also known as spectral determinant) of a particular random matrix H. In particular, the total number of stationary points in the whole space is given by

$$N_{\mathrm{tot}}(\mu) = \frac{1}{\mu^N} \langle |\det(\mu I_N - H)| \rangle. \tag{4}$$

Whenever the physical problem necessitates dealing with the absolute value of the determinant, its presence is considered to be a serious technical challenge; see [9], and references therein. In particular, intensive work and controversy persist in calculating the so-called *thermodynamic* complexity of the *free* energy for the standard Sherrington-Kirkpatrick model of spin glasses [6] or its generalizations [5]. Several heuristic schemes based on various versions of the replica trick were proposed in the literature recently to deal with the problem; see discussion and further references in [6]. Despite some important insights, the present status of the methods is not yet completely satisfactory.

In the present Letter we propose two different methods of dealing with the modulus of determinants, both free from any mathematical uncertainty. The first method is specific for the problem in hand and is heavily based on the isotropy of the correlation function of the random field V in Eq. (2). Exploitation of this fact provides one with a possibility to employ, after some manipulations, the standard methods of the random matrix theory [10] and find the explicit expression for the number N_{tot} of stationary points for any spatial dimension N.

To follow such a route we notice that the statistical properties of the potential V result in the following second-order moments of the entries H_{ij} $\{(i,j) = 1, \ldots, N\}$:

$$\langle H_{il} H_{jm} \rangle = \frac{J^2}{N} [\delta_{ij}\delta_{lm} + \delta_{im}\delta_{lj} + \delta_{il}\delta_{jm}], \tag{5}$$

where we denoted $J^2 = f''(0)$. This allows one to write down the density of the joint probability distribution (JPD) of the matrix H explicitly as

$$\mathcal{P}(H)dH \propto dH \exp\left\{-\frac{N}{4J^2}\left[\mathrm{Tr}(H^2) - \frac{1}{N+2}(\mathrm{Tr}H)^2\right]\right\}, \tag{6}$$

where $dH = \prod_{1 \le i \le j \le N} dH_{ij}$ and the proportionality constant can be easily found from the normalization condition and will be specified later on. It is evident that such a JPD is invariant with respect to rotations $H \to O^{-1}HO$ by orthogonal matrices $O \in O(N)$, but it is nevertheless different from the standard one typical for the so-called Gaussian orthogonal ensemble (GOE) [10]. However, introducing one extra Gaussian integration it is in fact straightforward to relate averaging over the JPD (6) to that over the standard GOE. In particular,

$$\langle |\det(\mu I_N - H)| \rangle = \int_{-\infty}^{\infty} \frac{dt}{\sqrt{2\pi}} e^{-N(t^2/2)} \langle |\det[(\mu + Jt)I_N - H_0]| \rangle_{\mathrm{GOE}}, \tag{7}$$

where the averaging over H_0 is performed with the GOE-type measure: $dH_0\, C_N \exp\{-(N/4J^2)\mathrm{Tr}H_0^2\}$, with $C_N = N^{1/2}/[(2\pi J^2/N)^{N(N+1)/4} 2^{N/2}]$ being the relevant normalization constant.

To evaluate the ensemble averaging in (7) in the most economic way one can exploit explicitly the mentioned rotational $O(N)$ invariance, and at the first step in a standard way [10] reduce the ensemble averaging to the integration over eigenvalues $\lambda_1, \ldots, \lambda_N$ of the matrix H_0. After a convenient rescaling $\lambda_i \to J\sqrt{2/N}\lambda_i$ the resulting expression acquires the form

$$\langle |\det[(\mu + Jt)I_N - H_0]| \rangle_{\mathrm{GOE}} \propto \int_{-\infty}^{\infty} d\lambda_1 \ldots \int_{-\infty}^{\infty} d\lambda_N \times \prod_{i<j}^{N} |\lambda_i - \lambda_j| \prod_{i=1}^{N} |\sqrt{N/2}(m+t) - \lambda_i| e^{-(1/2)\lambda_i^2}, \tag{8}$$

where we denoted $m = \mu/J$. One may notice that the above N-fold integral can be further rewritten as an $N+1$ fold integral:

$$e^{(N/4)(m+t)^2} \int_{-\infty}^{\infty} d\lambda_1 \ldots \int_{-\infty}^{\infty} d\lambda_{N+1} \prod_{i=1}^{N+1} e^{-(1/2)\lambda_i^2} \times \delta[\sqrt{N/2}(m+t) - \lambda_{N+1}] \prod_{i<j}^{N+1} |\lambda_i - \lambda_j|.$$

Such a representation makes it immediately evident that, in fact, the expectation value of the modulus of the determinant in question is simply proportional to the mean spectral density $\nu_{N+1}[m+t]$ [also known as one-point correlation function $R_1^{(N+1)}[\sqrt{N/2}(m+t)]$; see [10]] of the same GOE matrix H_0 but of enhanced size $(N+1) \times (N+1)$:

$$\langle|\det[(\mu + Jt)I_N - H_0]|\rangle_{\rm GOE} \propto e^{(N/4)(m+t)^2}\nu_{N+1}[(m+t)],$$
$$\nu_N[\lambda] = \frac{1}{N}\langle {\rm Tr}\,\delta(\lambda I_N - H_0)\rangle_{\rm GOE}. \quad (9)$$

The last relation provides the complete solution of our original problem for any value of N, since the one-point function $R_1^{(N+1)}[x]$ is known in a closed form [10] for any value of N in terms of the Hermite polynomials $H_k(x)$. In particular, for any odd integer N we have

$$\langle|\det[(\mu + Jt)I_N - H_0]|\rangle_{\rm GOE} = \frac{J^N}{\sqrt{2\pi}}\left[\left(\frac{N-1}{2}\right)!\right]\left[e^{-(x^2/2)}\sum_{k=0}^{N}\frac{1}{2^k k!}H_k^2(x) + \frac{1}{2^{N+1}N!}H_N(x) \times \int_{-\infty}^{\infty} e^{-(u^2/2)}H_{N+1}(u){\rm sgn}(x-u)\,du\right], \quad (10)$$

where we denoted $x = \sqrt{N/2}(m+t)$ for brevity. For even integer N one more term arises; see [10].

Being interested mainly in extracting the complexity $\Sigma(\mu) = \lim_{N\to\infty} N^{-1}\ln N_{\rm tot}(\mu)$ we have to perform an asymptotic analysis of Eqs. (4), (7), (9), and (10). In principle, one can employ the known large-N asymptotics of the Hermite polynomials, but we find it more convenient to use an alternative, well-known representation for the mean eigenvalue density:

$$\nu_N[\lambda] = \frac{1}{N\pi}{\rm Im}\frac{\partial}{\partial\lambda_b}\left\langle\frac{\det(\lambda I_N - H_0)}{\det(\lambda_b I_N - H_0)}\right\rangle\Bigg|_{\lambda_b=\lambda}. \quad (11)$$

The ensemble average of the ratio of the two determinants can be easily found in the framework of the supersymmetric approach [11]. Following a variant of this method we use for our analysis the following integral representation for the derivative of the ratio of two determinants featured in Eq. (11) [see Eq. (46) in Ref. [12]]:

$$\nu_N[\lambda] \propto {\rm Re}\int_{-\infty}^{\infty}\frac{dq}{q^2}e^{-N\mathcal{L}(q)}G_N(q,\lambda), \quad (12)$$

where $\mathcal{L}(q) = (q^2/2) + i\lambda q - \ln(q)$ and

$$G_N(q) = \int_0^\infty dp_1\frac{(p_1-q)}{p_1^{3/2}}\exp\left(-\frac{N}{2}\mathcal{L}(p_1)\right)\times\int_0^\infty dp_2\frac{(p_2-q)}{p_2^{3/2}}|p_1^2 - p_2^2|\exp\left(-\frac{N}{2}\mathcal{L}(p_2)\right). \quad (13)$$

The form of the above expressions Eqs. (12) and (13), suggests that the large-N asymptotics should be given by a saddle-point contribution in all integration variables. It turns out, however, that the situation is not that simple. To perform the asymptotic analysis accurately it is convenient first to get rid of the nonanalyticity in the integrand by passing in the last expression to new variables $0 \le r < \infty$, $-\infty < \theta < \infty$ by $p_1 = re^{\theta}$, $p_2 = re^{-\theta}$. Further introducing $u = r(\cosh\theta - 1)$ we arrive at

$$G_N(q) \propto \int_0^\infty\frac{dr}{r^2}\exp\{-N\mathcal{L}(r)\}\times\int_0^\infty du(r+u)[(r-q)^2 - 2qu]e^{-N[u^2+2u(r+i\lambda/2)]}. \quad (14)$$

The saddle-point value in the u variable is obviously $u_s = -(r + i\lambda/2)$, but it can yield no contribution as long as ${\rm Re}\, r > 0$ along the contour of integration. The analysis reveals that this is indeed the case for $|\lambda| < 2$. Under that condition the u integral is dominated by the vicinity of its end point $u = 0$, whereas integrals over r and q are instead saddle-point dominated, the dominant saddle points being $r_s = (-i\lambda + \sqrt{4-\lambda^2})/2$ and $q_s = (-i\lambda - \sqrt{4-\lambda^2})/2$. Calculating the corresponding contribution we arrive, as expected, to the standard semicircular spectral density $\nu[\lambda] = (1/2\pi)\sqrt{4-\lambda^2}$. If, however, the parameter λ is such that $|\lambda| > 2$, the situation turns out to be very different. In that case both saddle-point values $r_s = i(-\lambda \pm \sqrt{\lambda^2 - 4})/2$ are purely imaginary, necessitating a part of the steepest descent contour to be chosen along the imaginary axis ${\rm Re}\, r = 0$. As a result, an additional contribution from the saddle-point u_s turns out to be operative. Although such a contribution is exponentially small in comparison with one dominated by the vicinity of $u = 0$, it is the only one which survives after taking Re in (12). Taking into account the saddle pont u_s induces modifications of the relevant exponential term in the r variable, which now becomes $\exp\{N[(r^2/2) + \ln r - (\lambda^2/4)]\}$ and replaces the former expression $\exp\{-N\mathcal{L}(r)\}$. The relevant saddle point for r then turns out to be $r_s = -i$ as long as $\lambda > 2$, and it results in exponentially small (''instanton'') value for the spectral density:

$$\nu[\lambda] \propto \exp\left\{-N\left[\frac{\lambda^2}{4} - \ln\frac{\lambda + \sqrt{\lambda^2-4}}{2}\right]\right\}, \qquad \lambda > 2, \quad (15)$$

where we only kept factors relevant for calculating the complexity in the limit of large N.

Finally, we employ the relation (9) between the mean spectral density and the expectation value of the modulus of the spectral determinant for GOE matrices, and substitute the resulting expression into the integral (7). In the latter we can again exploit the saddle-point method for asymptotic analysis. For $0 < m < 1$ the relevant saddle point is $t_s = m$ satisfying $0 < \lambda_s = t_s + m < 2$, and validating the use of the semicircular spectral density $\nu[\lambda_s] = (1/2\pi)\sqrt{4-\lambda_s^2}$ in the calculation. This yields

$$\langle|\det(\mu I_N - H)|\rangle \propto e^{N/2\,(m^2-1)}\sqrt{1-m^2}, \qquad 0 < m < 1. \tag{16}$$

For $m > 1$, however, it turns out that one has to use Eq. (15) for the spectral density. The corresponding saddle-point value t_s in the t integral is given by the solution of the equation $m = \frac{1}{2}(\lambda_s + \sqrt{\lambda_s^2 - 4})$ for the variable $\lambda_s = t_s + m$. The solution is easily found to be simply $\lambda_s = m + m^{-1}$ (note that $\lambda_s > 2$, ensuring consistency of the procedure) which yields the resulting value for the modulus of the determinant to be given by

$$\langle|\det(\mu I_N - H)|\rangle \propto e^{N\ln m}, \qquad m > 1. \tag{17}$$

Invoking our basic relation Eq. (4) for $N_{\rm tot}$ we see that the landscape complexity $\Sigma(\mu)$ of the random potential function (1) is given by

$$\Sigma(\mu) = \frac{1}{2}\left(\frac{\mu^2}{J^2} - 1\right) - \ln(\mu/J), \qquad \mu < \mu_c = J, \tag{18}$$

$$\Sigma(\mu) = 0, \qquad \mu > \mu_c = J. \tag{19}$$

Earlier works referred to the critical value $\mu_c = J$ as, on one hand, signaling the onset of a nontrivial glassy dynamics [8], and, on the other hand, corresponding to the point of a breakdown of the replica-symmetric solution [7]. Our calculation provides an independent support to the point of view attributing both phenomena to extensive number of stationary points in the energy landscape. At the critical value the complexity vanishes quadratically: $\Sigma(\mu \to \mu_c) \propto (\mu_c - \mu)^2/\mu_c^2$.

Finally, let us very shortly discuss an alternative, less model-specific technique of evaluating the absolute value of the spectral determinant. It is based on the following useful identity (see, e.g., [13]):

$$|\det(\mu I_N - H)| = \lim_{\epsilon\to 0}\frac{[\det(\mu I_N - H)]^2}{\sqrt{\det((\mu - i\epsilon)I_N - H)}\sqrt{\det((\mu + i\epsilon)I_N - H)}} \tag{20}$$

valid for any matrix H with purely real eigenvalues. For the particular case of real symmetric matrices H one can represent the two factors in the denominator of the right-hand side in terms of the Gaussian integrals absolutely convergent as long as $\epsilon > 0$. Further representing the determinantal factors in the numerator in terms of the Gaussian integral over anticommuting (Grassmann) variables we thus get a *bona fide* supersymmetric [11] object to be analyzed. Simultaneous presence in the starting expression both $\mu^{\pm} = \mu \pm i\epsilon$ and μ makes the calculation in this case more involved in comparison with just a simple ratio of two determinants, as in (11). It is nevertheless an important fact that the possibility to perform the ensemble average explicitly exists whenever matrix entries of H are Gaussian distributed, not requiring any matrix invariance or even independence of the matrix entries. A similar strategy may be even employed when H is a stochastic differential operator with a certain Gaussian part, as in the notoriously difficult case of the random field Ising model [14]. For this reason it is natural to expect that the suggested method could be helpful beyond the present model, e.g., when discussing free energy landscapes for spin-glass related problems [5,6].

In summary, we calculated the mean total number $N_{\rm tot}$ of stationary points for a N-dimensional potential consisting of a quadratic well of strength μ and of a random Gaussian piece V. In particular, for $N \to \infty$ we found that the potential is characterized by a finite landscape complexity: $N_{\rm tot} \sim e^{N\Sigma}$, $\Sigma > 0$, as long as $\mu < \mu_c$, and for $\mu \to \mu_c$ the complexity Σ vanishes quadratically. Finally, we discuss a general method of calculating the mean absolute value of the spectral determinant.

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