Exact Maximal Height Distribution of Fluctuating Interfaces

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We present an exact solution for the distribution $P(h_m, L)$ of the maximal height h_m (measured with respect to the average spatial height) in the steady state of a fluctuating Edwards-Wilkinson interface in a one dimensional system of size L with both periodic and free boundary conditions. For the periodic case, we show that $P(h_m, L) = L^{-1/2} f(h_m L^{-1/2})$ for all $L > 0$, where the function $f(x)$ is the Airy distribution function that describes the probability density of the area under a Brownian excursion over a unit interval. For the free boundary case, the same scaling holds, but the scaling function is different from that of the periodic case. Numerical simulations are in excellent agreement with our analytical results. Our results provide an exactly solvable case for the distribution of extremum of a set of *strongly correlated* random variables.

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Fluctuating interfaces are among the most well studied nonequilibrium systems due to their simplicity as well as numerous practical applications in systems such as growing crystals, molecular beam epitaxy, fluctuating steps on metals, and growing bacterial colonies [1]. While the past studies mostly focused on the scaling properties of the surface roughness characterized by the average width of the surface height [1], the more recent theoretical and experimental studies have dealt with a variety of other important characteristics of a fluctuating interface. These include the distribution of the width of heights in the steady state [2], the statistics of first-passage events or persistence [3,4], the density of local maxima or minima of heights [5], the distribution of the spatially averaged height [6], as well as the distribution of height at any fixed point in space [7] in growing one dimensional Kardar-Parisi-Zhang (KPZ) interfaces [8], the cycling effects [9], the distribution of extremal Fourier intensities [10], etc.

Recently Raychaudhuri *et al.* [11] studied a different characteristic, namely, the global maximal relative height (MRH) (measured with respect to the spatially averaged growing height) of a fluctuating interface. This is an important observable for two principal reasons. First, it has important technological significance such as in batteries where a short circuit occurs when the highest point of a metal surface on one electrode reaches the opposite one [11]. Second, the maximal height is an extreme observable measuring a rare event. While the extreme value statistics is well understood for a set of *independent* random variables [12], only recently physicists have been paying attention to the distribution of the extremum of a set of *correlated* random variables, as this question is appearing increasingly frequently in a number of problems ranging from disordered systems [13] to various problems in computer science such as growing search trees[14] and networks [15]. In a fluctuating interface, the heights are strongly correlated and hence a knowledge of the distribution of their maximum (or minimum) would provide important insights into this important general class of extreme value problems where the random variables are correlated.

In Ref. [11], the authors argued quite generally that the MRH h_m of an interface in its stationary state in a finite system of size L scales as the roughness of the surface, $h_m \sim L^{\alpha}$ for large L, where α is the roughness exponent. This indicates that the normalized probability density function (PDF) of h_m has a scaling form, $P(h_m, L) \sim L^{-\alpha} f(h_m/L^{\alpha})$. This was demonstrated numerically in [11] for a one dimensional lattice model belonging to the Edwards-Wilkinson (EW) universality class [16], where $\alpha = 1/2$. Further, it was argued that the scaling function $f(x)$ is sensitive to the boundary conditions [11].

In this Letter, using simple path integral techniques we present an exact solution of the scaling function $f(x)$ for the one dimensional EW model, both for the periodic and the free boundary conditions. For the periodic boundary case, we show that the scaling function $f(x)$ is the so-called Airy distribution function (not to be confused with the Airy function) which is the PDF of the area under a Brownian excursion over a unit interval and has been well studied in the mathematics literature [17–20]. We also calculate exactly the corresponding scaling function for the free boundary condition and show that it is different from the periodic case. All the moments of h_m are also computed exactly for both the boundary conditions. These results are in excellent agreement with the simulation results obtained by the numerical integration of the discretized 1D EW equation. Our results thus provide an exactly solvable case for the distribution of the extremum of a set of *strongly correlated* random variables.

Our starting point is the one dimensional EW model [16] which prescribes a linear evolution equation for the height $H(x, t)$,

$$\frac{\partial H(x,t)}{\partial t} = \frac{\partial^2 H(x,t)}{\partial x^2} + \eta(x,t), \tag{1}$$

where $\eta(x,t)$ is a Gaussian white noise with zero mean and a correlator, $\langle \eta(x,t)\eta(x',t')\rangle = 2\delta(x-x')\delta(t-t')$. The Eq. (1) has a soft (zero wave vector) mode since the spatially averaged height $\overline{H(x,t)} = \int_0^L H(x,t)dx/L$ keeps on growing with time (typically as $\sqrt{t/L}$) even in a finite system of size L. Hence, it is useful to subtract this zero mode from the height and define the relative height, $h(x,t) = H(x,t) - \overline{H(x,t)}$, whose distribution then reaches a stationary state in the long time limit in a finite system. Note that, by definition,

$$\int_0^L h(x,t)dx = 0. \tag{2}$$

We see later that this constraint of zero total area under the relative height h plays an important role in determining the MRH distribution. All the other nonzero modes of h evolve identically as those of the actual height H.

We first consider the periodic boundary condition, $h(0) = h(L)$. In this case, one can decompose the relative height $h(x,t)$ into a Fourier series, $h(x,t) = \sum_{m=-\infty}^{\infty} \tilde{h}(m,t)e^{2\pi i m x/L}$. Substituting this in Eq. (1), one finds that different nonzero Fourier modes decouple from each other and one can easily calculate any correlation function. In particular, it is easy to see that the height $h(x,t)$ at any given point converges to a stationary Gaussian distribution as $t \to \infty$, $P_{\rm st}(h) = e^{-h^2/2w^2}/\sqrt{2\pi w^2}$, where the width $w(L) = \sqrt{\langle h^2\rangle} = \sqrt{L/12}$ for all L. Moreover, one can also show that $\langle \partial_x h \partial_{x'} h\rangle \to \delta(x-x') - 1/L$ in the stationary state. The local slopes $\partial_x h$ are thus uncorrelated except for the overall constraint due to the periodic boundary condition, $\int_0^L dx \partial_x h = 0$ that gives rise to the residual $1/L$ term. These facts can be collected together to write the joint probability distribution of the heights (multivariate Gaussian distribution) in the stationary state as

$$P[\{h\}] = C(L)e^{-1/2\int_0^L d\tau(\partial_\tau h)^2} \times \delta[h(0)-h(L)]\delta\Big[\int_0^L h(\tau)d\tau\Big], \tag{3}$$

where $C(L)$ is a normalization constant and the two delta functions take care, respectively, of the periodic boundary condition and the zero area constraint in Eq. (2). The constant $C(L) = \sqrt{2\pi}L^{3/2}$ can be evaluated exactly by integrating Eq. (3) over all heights and setting it to unity [21]. One can check that if one integrates out all the heights in Eq. (3) except at one point, one recovers the single point stationary height distribution mentioned before.

We next define the cumulative distribution of the MRH, $F(h_m, L) = \text{Prob}[\max\{h\} < h_m, L]$. The PDF of the MRH is simply the derivative, $P(h_m, L) = \frac{\partial F(h_m,L)}{\partial h_m}$. Clearly $F(h_m, L)$ is also the probability that the heights at all points in $[0, L]$ are less than h_m and can be formally written using the measure in Eq. (3) as a path integral,

$$F(h_m,L) = C(L)\int_{-\infty}^{h_m} du \int_{h(0)=u}^{h(L)=u} \mathcal{D}h(\tau)e^{-1/2\int_0^L d\tau(\partial_\tau h)^2} \times \delta\Big[\int_0^L h(\tau)d\tau\Big] I(h_m,L), \tag{4}$$

where $I(h_m, L) = \prod_{\tau=0}^{L}\theta[h_m - h(\tau)]$ is an indicator function which is 1 if all the heights are less than h_m and zero otherwise. All the paths inside the path integral propagate from its initial value $h(0) = u$ to its final value $h(L) = u$, where $u \le h_m$ (since by definition h_m is the maximum). A change of variable, $y(\tau) = h_m - h(\tau)$ and $v = h_m - u$, in the path integral in Eq. (4) gives

$$F(h_m,L) = C(L)\int_0^{\infty} dv \int_{y(0)=v}^{y(L)=v} \mathcal{D}y(\tau)e^{-1/2\int_0^L d\tau(\partial_\tau y)^2} \times \delta\Big[\int_0^L y(\tau)d\tau - A\Big] I(h_m,L), \tag{5}$$

where $I(h_m, L) = \prod_{\tau=0}^{L}\theta[y(\tau)]$ and $A = h_m L$. Note that h_m appears only through A in the delta function, and hence $F(h_m, L) = \mathcal{F}(A, L)$. In subsequent calculations, we keep a general A in Eq. (5) and finally use $A = h_m L$. Next we take the Laplace transform with respect to A in Eq. (5) and identify the quantity inside the exponential as the action corresponding to a single particle quantum Hamiltonian, $\hat{H} \equiv -\frac{1}{2}\frac{\partial^2}{\partial y^2} + V(y)$, where $V(y) = \lambda y$ for $y > 0$ and $V(y) = \infty$ for $y \le 0$. The latter condition takes care of the indicator function. Using the standard bra-ket notation we get

$$\int_0^{\infty} \mathcal{F}(A,L)e^{-\lambda A}dA = C(L)\int_0^{\infty} dv\langle v|e^{-\hat{H}L}|v\rangle = C(L)\text{Tr}[e^{-\hat{H}L}], \tag{6}$$

where Tr is the trace. Thus our problem is reduced to calculating just the eigenvalues of the above Hamiltonian $\hat{H}$ which has only bound states and hence discrete eigenvalues. Solving the Schrödinger equation, one finds that the wave function (up to a normalization constant) is simply $\psi_E(y) = \text{Ai}[(2\lambda)^{1/3}(y - E/\lambda)]$, where $\text{Ai}(z)$ is the standard Airy function [22]. This wave function must vanish at $y = 0$ which determines the discrete eigenvalues, $E_k = \alpha_k \lambda^{2/3}2^{-1/3}$ for $k = 1, 2, \ldots$, where α_k's are the magnitude of the zeros of $\text{Ai}(z)$ on the negative real axis. For example, one has $\alpha_1 = 2.3381\ldots$, $\alpha_2 = 4.0879\ldots$, $\alpha_3 = 5.5205\ldots$, etc. Upon formally inverting the Laplace transform in Eq. (6) and putting $A = h_m L$, we find

$$F(h_m,L) = \sqrt{2\pi}L^{3/2}\int_{-i\infty}^{+i\infty}\frac{d\lambda}{2\pi i}e^{\lambda h_m L}\sum_{k=1}^{\infty}e^{-\alpha_k\lambda^{2/3}2^{-1/3}L}. \tag{7}$$

Taking derivative with respect to h_m in Eq. (7) and making a change of variable, $\lambda = sL^{-3/2}$, we arrive at our

main result, $P(h_m, L) = L^{-1/2} f(h_m L^{-1/2})$ for all L, where the Laplace transform of $f(x)$ is given by

$$\int_0^\infty f(x) e^{-sx} dx = s\sqrt{2\pi} \sum_{k=1}^{\infty} e^{-\alpha_k s^{2/3} 2^{-1/3}}. \qquad (8)$$

Interestingly, the right-hand side (rhs) of Eq. (8) turns out precisely to be the Laplace transform of the PDF of the area under a Brownian excursion over a unit interval [19]. A Brownian excursion over the interval $[0, 1]$ is simply a Brownian motion pinned at zero at the two ends of the interval and conditioned to stay positive in between. Inverting the Laplace transform in Eq. (8) one obtains $f(x)$, known as the Airy distribution function [19],

$$f(x) = \frac{2\sqrt{6}}{x^{10/3}} \sum_{k=1}^{\infty} e^{-b_k/x^2} b_k^{2/3} U(-5/6, 4/3, b_k/x^2), \qquad (9)$$

where $b_k = 2\alpha_k^3/27$ and $U(a, b, z)$ is the confluent hypergeometric function [22]. In Fig. 1, we have plotted $f(x)$ in Eq. (9) using the Mathematica and compared it with the numerical scaling function generated by collapsing the data for three different system sizes obtained by numerically integrating the space-time discretized form of Eq. (1). Evidently the agreement is very good.

It is easy to obtain the small x behavior of x from Eq. (9), since only the $k = 1$ term dominates as $x \to 0$. Using $U(a, b, z) \sim z^{-a}$ for large z, we get as $x \to 0$,

$$f(x) \to \frac{8}{81} \alpha_1^{9/2} x^{-5} \exp\left[-\frac{2\alpha_1^3}{27x^2} \right]. \qquad (10)$$

This essential singular tail near $x \to 0$ was conjectured in [11] based on numerics, though the exact form was missing. The asymptotic behavior at large x is more tricky to derive [23] from Eq. (9). Even the calculation of moments from Eq. (8) is rather nontrivial. However, it is possible to write down an exact recursion relation for the moments [18,19] and using these results, we get $\langle h_m^n \rangle = M_n L^{n/2}$, where $M_0 = 1$, $M_1 = \sqrt{\pi/8}$, $M_2 = 5/12$, $M_3 = 15\sqrt{\pi}/64\sqrt{2}$, $M_4 = 221/1008$, etc. Only the second moment $\langle h_m^2 \rangle = 5L/12$ was computed before in [11] by using a different method. Finally, one finds that for large n, $M_n \sim (n/12e)^{n/2}$. Substituting an anticipated large x decay of the form, $f(x) \sim \exp[-ax^b]$ in $M_n = \int_0^\infty f(x) x^n dx$, we get $M_n \sim (n/abe)^{n/b}$ for large n. Comparing this with the exact large n behavior of M_n we get $a = 6$ and $b = 2$, indicating $f(x) \sim \exp[-6x^2]$ as $x \to \infty$.

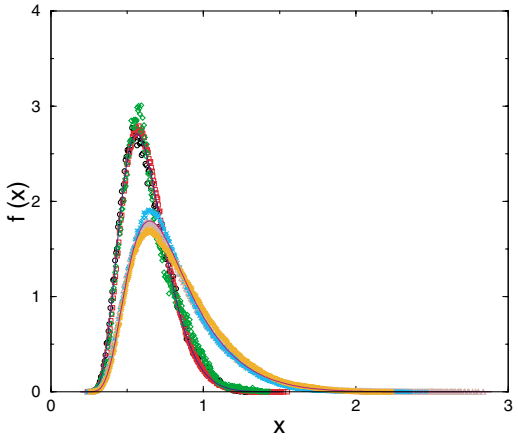

FIG. 1 (color online). The scaling function $f(x)$ for the MRH distribution for both the periodic (the top four curves) and the free (the lower four curves) boundary conditions. In both cases, the numerical curves (shown by symbols) are obtained by collapsing the data from the numerical integration of the space-time discretized form of the EW Eq. (1) for three system sizes $L = 256$, $L = 384$, and $L = 512$. They are compared to the corresponding analytical scaling functions (solid lines).

There is an alternative elegant probabilistic derivation of the above result which we outline briefly. It proceeds by establishing the equivalence

$$h(x) \equiv B(x) - \frac{1}{L} \int_0^L B(\tau) d\tau, \qquad (11)$$

where $h(x)$ is the stationary EW interface with periodic boundary condition, $B(x)$ is a Brownian bridge [a Brownian motion such that $B(0) = B(L) = 0$] and $\equiv$ means that the left-hand side (lhs) has the same probability distribution as the rhs. First, by construction the rhs satisfies the area constraint in Eq. (2). Second, both the lhs and the rhs of Eq. (11) are Gaussian variables and hence to establish the equivalence in Eq. (11), it is sufficient to show that their respective two-point correlators are identical. For example, one finds[21] from Eq. (1) that in the stationary state, $\langle h(x) h(x') \rangle = [L^2/6 - L|x - x'| + (x - x')^2]/2L$ for all L. Similarly, one can calculate the two-point correlator of the rhs using the representation $B(\tau) = x(\tau) - \tau x(L)/L$, where $x(\tau)$ is ordinary Brownian motion starting at $x(0) = 0$ and with a correlator, $\langle x(\tau) x(\tau') \rangle = \min(\tau, \tau')$. This representation guarantees that $B(0) = B(L) = 0$. We find that the two-point correlator of the rhs is exactly the same as $\langle h(x) h(x') \rangle$. This establishes the equivalence in law in Eq. (11) rigorously. Hence, the maximum of $h(x)$ has the same distribution as the maximum of the rhs of Eq. (11) which, incidentally, was computed by Darling in the context of statistical data analysis and he found [17] exactly the same Laplace transform as in Eq. (8).

We next consider the free boundary condition where the two ends of the interface are held free. In this case, the joint distribution of heights in the stationary state is given by the same formula as in Eq. (3), except without the delta function $\delta[h(0) - h(L)]$. This changes the normalization constant to $C(L) = L$. However, unlike the simple trace in the periodic case in Eq. (6), the Laplace transform in the free case turns out to be more complicated [21]. Omitting details, we find the same scaling as in the periodic case, $P(h_m, L) = L^{-1/2} f(h_m L^{-1/2})$, though the scaling function has a different Laplace transform $\tilde{f}(s) = \int_0^\infty f(x) s^{-sx} dx$,

$$\tilde{f}(s) = s^{2/3}2^{-1/3}\sum_{k=1}^{\infty} C(\alpha_k)e^{-\alpha_k s^{2/3}2^{-1/3}}, \qquad (12)$$

where $C(\alpha) = [\int_{-\alpha}^{\infty} \mathrm{Ai}(z)dz]^2/[\mathrm{Ai}'(-\alpha)]^2$ and $\mathrm{Ai}'(z) = dA_i(z)/dz$. This Laplace transform does not seem to have appeared before in the mathematics literature. One can again express the function $f(x)$ in terms of the confluent hypergeometric function [21], a Mathematica plot of which is shown in Fig. 1 that matches well with the numerical simulations. The small x behavior can again be found easily and we get

$$f(x) \to \frac{2\sqrt{2}}{27\sqrt{\pi}} C(\alpha_1)\alpha_1^{7/2} x^{-4} \exp\left[-\frac{2\alpha_1^3}{27x^2}\right], \qquad (13)$$

where $C(\alpha_1) = 3.302\,78\ldots$, evaluated using the Mathematica. Thus the function $f(x)$ decays slightly faster as $x \to 0$ compared to the periodic case in Eq. (10). We also calculated the moments exactly [21], $\langle h_m^n \rangle = \mu_n L^{n/2}$, where $\mu_0 = 1$, $\mu_1 = \sqrt{2/\pi}$, $\mu_2 = 17/24$, $\mu_3 = 123\sqrt{2}/140\sqrt{\pi}$, etc. We found that for large n, $\mu_n \sim [n/3e]^{n/2}$ which provides the asymptotic large x tail of $f(x)$, $f(x) \sim \exp[-3x^2/2]$ that falls off less rapidly than the periodic case where $f(x) \sim \exp[-6x^2]$.

In summary, we have presented exact analytical results for the maximal height distribution of the 1D EW interface in its steady state, with different boundary conditions. For the periodic boundary condition, our result explains the numerical results of Ref. [11]. But our method also predicts new results for other boundary conditions. Besides, our result is robust and holds even for 1D KPZ equation since it has the same steady state measure as the EW equation in one dimension. We have verified this theoretical prediction numerically [21]. Also, our path integral technique is quite general and can, in principle, be used to calculate other physically interesting quantities, such as the distribution of the difference between the maximum and the minimum heights of a surface. It would be interesting to extend our method to higher dimensions where little is known about the maximal height of a growing surface.

Finally, many experimental systems are well described by the 1D EW equation (1). Examples include, among others, the high-temperature step fluctuations in Si(111)-Al surfaces[4,24] and the displacements of nonmagnetic particles in dipolar chains at low magnetic field[25]. Besides, the displacements of beads in a polymer chain with harmonic interaction (the Rouse model [26]) also evolve via the 1D EW equation. Thus our results are relevant in these systems and hopefully the MRH distribution can be measured experimentally in such systems.

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