Einstein Gravity on the Codimension 2 Brane?

Paul Bostock,[1] Ruth Gregory,[1,2] Ignacio Navarro,[2] and Jose Santiago[2]

[1]*Center for Particle Theory, Department of Mathematical Sciences, University of Durham, DH1 3LE Durham, United Kingdom*

[2]*Institute for Particle Physics Phenomenology, Department of Physics, University of Durham, DH1 3LE Durham, United Kingdom*

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We look at general brane worlds in six-dimensional Einstein-Gauss-Bonnet gravity. We find the general matching conditions for the brane world, which remarkably give precisely the four-dimensional Einstein equations for the brane, even when the extra dimensions are noncompact and have infinite volume. Relaxing regularity of the curvature in the vicinity of the brane, or having a thick brane, gives rise to an additional term containing information on the brane's embedding in the bulk. We comment on the relevance of these results to a possible solution of the cosmological constant problem.

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The brane world paradigm, or the idea that our universe might be a slice of some higher dimensional spacetime, has proved a compelling alternative to standard Kaluza-Klein (KK) methods of having more than four dimensions. Briefly, in contrast to KK *compactifications*, which have small and compact extra dimensions, brane worlds can have large, even noncompact, extra dimensions, which have potentially important experimental consequences [1,2]. We do not directly see the extra dimensions since we are confined to our brane world; rather, their presence is felt via short-scale corrections to Newton's law, in some cases large scale modifications of gravity, and as a means of generating the hierarchy between the weak and Planck scales. Although being confined to a slice in spacetime might seem odd, such confinement is in fact a common occurrence. The early brane world scenarios [3], for example, used zero modes on topological defects, and in string theory we have confinement of gauge theories on D-branes.

Formally, the brane world is a submanifold of the spacetime manifold and can have any number of codimensions—the number of extra dimensions—up to a maximum of 6/7 for string/M-theory. By far, the best investigated and understood brane world scenario is the codimension 1 case, or a toy five-dimensional example motivated by the Horava-Witten compactification of M-theory [4]. This range of models, based on the seminal work of Randall and Sundrum (RS) [2], has all the features one requires: Einstein gravity at some scale with calculable modifications, well-defined cosmology asymptoting standard cosmology at late times, and has the additional allure of exhibiting directly aspects of the gravity/gauge theory (adS/CFT) correspondence.

Far less well understood are higher codimension brane worlds. Although the pioneering work on resolving the hierarchy problem took place within the context of higher codimension, empirical models lack the gravitational consistency of the RS scenarios. Attempts to include self-gravity have met with some success in codimension 2 [5–7], but for codimension 3 or higher, the situation seems to be more problematic [8,9].

Codimension 2 brane worlds offer also some interesting properties that can be exploited to attack the cosmological constant problem [7], but one drawback is that, in contrast to codimension 1, we appear to be very restricted in our allowed brane energy momenta. Typically, a brane in its ground state has a very special energy-momentum tensor, which is isotropic and has the property that energy = tension. If we wish to have any matter on the brane, then we require a varying energy to tension ratio. However, as pointed out by Cline *et al.* [10] for cosmological branes, this is inconsistent with some basic minimal assumptions about the nature of the brane world. Namely, it causes singularities in the metric around the brane world, necessitating the introduction of a cutoff and hence introducing questions of model dependence.

In this Letter we suggest that the solution to the apparent sterility of codimension 2 brane worlds might lie in the Gauss-Bonnet term. This is a term that can be added to the action in $D > 4$ (it is a topological invariant in 4D), which is quadratic in the curvature tensor but has the well-known property that the equations of motion derived from it remain second order differential equations for the metric. In fact, since $\mathcal{O}(R^2)$ corrections to the Einstein-Hilbert Lagrangian do arise in the low energy limit of string theory, the inclusion of this type of term could be regarded as mandatory if one wants to embed any brane world solution into string/M-theory. Fluctuations around a flat background for this model were studied in [11], and the conclusions obtained are compatible with the ones presented in this Letter at the linearized level.

In trying to derive effective Einstein equations on the brane, it is worth comparing and contrasting with codimension 1. Recall that for codimension 1 there is a single normal to the brane world, hence a single direction from the brane world. For a general submanifold of codimension 2, there are now *two* normals, and for a regular submanifold we again have a well-defined coordinate patch around it defined by the Gauss-Codazzi formalism. (This method was used to derive effective actions of topological defects [12].) The problem is that

Gauss-Codazzi formalism requires some minimal regularity of the metric, and this is no longer the case for an infinitesimal brane world in codimension 2—the situation is even worse for codimension 3 and higher. Briefly, there is no well-defined "thin brane world" limit for the Einstein equations [13], or alternatively, for the conical deficit, it is not possible to put two normals at the location of the deficit, which have a well-defined inner product—it depends on whether you measure the outer or inner angle. In order to derive gravity on the brane therefore, we instead use a coordinate system that is defined in the vicinity of the brane world, and in which the effect of the brane formally appears as a delta function.

We assume that our brane world has a nonsingular metric, $\hat{g}_{\mu\nu}(x^\mu)$, which is continuous in the vicinity of the brane world. The coordinates x^μ label the brane world directions, and we will use greek indices to indicate brane world coordinates. We now take the set of points at a fixed proper distance from a particular x^μ on the brane, this will have topology S^1, and we label these points by x^μ, their proper distance, r, from x^μ, and an angle θ, which without loss of generality we will take to have the standard periodicity of 2π. This method provides a full coordinatization of spacetime in the vicinity of the brane world, and will be unique within the radii of curvature of the brane world. There are two remaining issues. One is that there is of course some ambiguity in the labeling of θ, which is equivalent to the choice of connection on the normal bundle of the brane world; however, for simplicity, we will assume that θ is chosen to make this connection vanish (in particular, this means we assume that the brane world is not self-intersecting). The second issue relates to the form of the bulk spacetime metric, which we will now assume has axial symmetry; i.e., ∂_θ is a Killing vector. This ansatz simplifies the bulk metric, and it is analogous to the assumption of Z_2 symmetry in the codimension 1 scenarios. The metric therefore can be seen in these coordinates to take the general form:

$$ds^2 = g_{\mu\nu}(x, r)dx^\mu dx^\nu - L^2(x, r)d\theta^2 - dr^2. \quad (1)$$

In order to obtain the brane world equations, we now expand the metric around the brane:

$$L(x, r) = \beta(x)r + O(r^2), \quad (2)$$

etc. For values of $\beta \neq 1$, we have a conical singularity at $r = 0$, which is interpreted as being due to a delta-function brane world source. Strictly speaking, at least in Einstein gravity, we cannot define a delta-function source in terms of a zero-thickness limit of finite sources [13]. Rather, we deduce the existence of the delta function in the Riemann tensor from the holonomy of a parallelly transported vector around the source. However, as the equations of motion make perfect sense with the delta function being encoded in a notional discontinuity of the radial derivatives of the metric at $r = 0$, we follow the standard procedure in this Letter of defining $L'(x, 0) = 1$, $g'_{\mu\nu}(x, 0) = 0$, in order to give rise to the required distributional behavior of the curvature in the gravitational equations (a prime denotes derivative with respect to r).

Therefore, for a general brane world, the problem we wish to solve is that of finding gravitating solutions that include the effect of a general brane energy-momentum tensor

$$T_{\rm MN} = \begin{pmatrix} \hat{T}_{\mu\nu}(x)\frac{\delta(r)}{2\pi L} & 0 \\ 0 & 0 \end{pmatrix} \quad (3)$$

(upper case latin indices run over all the dimensions). In particular, we will be interested in the relation between the 4D induced metric on the brane, $g_{\mu\nu}(x, 0) = \hat{g}_{\mu\nu}(x)$, and the brane energy-momentum tensor, $\hat{T}_{\mu\nu}(x)$. It is this relation which determines the nature of the gravitational interactions that a "brane observer" would measure.

Our starting point is the Einstein-Gauss-Bonnet (EGB) equation:

$$M_*^4(G_{\rm MN} + H_{\rm MN}) = T_{\rm MN} + S_{\rm MN}, \quad (4)$$

where

$$G_{\rm MN} = R_{\rm MN} - \frac{1}{2}g_{\rm MN}R, \quad (5)$$

and the Gauss-Bonnet contribution is given by

$$H_{\rm MN} = \alpha\Big[\frac{1}{2}g_{\rm MN}(R^2 - 4R^{PQ}R_{PQ} + R^{PQST}R_{PQST}) - 2RR_{\rm MN} + 4R_{MP}R^P_N + 4R^K_{MPN}R^P_K - 2R_{MQSP}R_N^{QSP}\Big], \quad (6)$$

with α a parameter with dimensions of (mass)$^{-2}$. $S_{\rm MN}$ is the bulk energy-momentum tensor, which we will not specify here other than to assume that it has no delta-function contributions.

If Eq. (4) is to be satisfied, there must be a singular contribution to the left-hand side (l.h.s.) of this equation with the structure $\sim \frac{\delta(r)}{L}$. As we have already discussed, such a contribution can arise from terms that contain

$$\frac{L''}{L} = -(1-\beta)\frac{\delta(r)}{L} + (\text{nonsingular part}), \quad (7)$$

$$\frac{\partial_r^2 g_{\mu\nu}}{L} = \partial_r g_{\mu\nu}\frac{\delta(r)}{L} + (\text{nonsingular part}). \quad (8)$$

In Einstein gravity, these latter terms are zero. However, since they could in principle be nonzero, we will retain them from now.

We must therefore set the delta-function contribution equal to the brane energy-momentum tensor in order to solve the equations of motion. After some calculation, one obtains that the only singular part of the l.h.s. of Eq. (4) lies in the μ, ν directions and is

$$-\frac{L''}{L}\Big[g_{\mu\nu} + 4\alpha\Big(R_{\mu\nu}(g) - \frac{1}{2}g_{\mu\nu}R(g)\Big)\Big] + \frac{\alpha}{L}\partial_r(L'W_{\mu\nu}), \quad (9)$$

where $W_{\mu\nu}$ is defined as the following combination of first derivatives of the four-dimensional metric:

$$W_{\mu\nu} = g^{\lambda\sigma}\partial_r g_{\mu\lambda}\partial_r g_{\nu\sigma} - g^{\lambda\sigma}\partial_r g_{\lambda\sigma}\partial_r g_{\mu\nu} + \frac{1}{2}g_{\mu\nu}[(g^{\lambda\sigma}\partial_r g_{\lambda\sigma})^2 - g^{\lambda\sigma}g^{\delta\rho}\partial_r g_{\lambda\delta}\partial_r g_{\sigma\rho}]. \tag{10}$$

We can now use the properties

$$-\frac{L''}{L} = (1-\beta)\frac{\delta(r)}{L} + \cdots, \tag{11}$$

$$\frac{\partial_r(L'W_{\mu\nu})}{L} = \beta W_{\mu\nu}|_{r=0^+}\frac{\delta(r)}{L} + \cdots, \tag{12}$$

to obtain the matching condition by equating the $\frac{\delta(r)}{L}$ terms of Eq. (4). This yields

$$2\pi(1-\beta)M_*^4\left[\hat{g}_{\mu\nu} + 4\alpha\hat{G}_{\mu\nu} + \alpha\frac{\beta}{1-\beta}\hat{W}_{\mu\nu}\right] = \hat{T}_{\mu\nu}, \tag{13}$$

where $\hat{G}_{\mu\nu}$ is the 4D Einstein tensor for the induced metric, $\hat{g}_{\mu\nu}$, and $\hat{W}_{\mu\nu} \equiv W_{\mu\nu}|_{r=0^+}$.

This is our main result: the gravitational equations of a brane world observer are the Einstein equations plus an extra Weyl term, $\hat{W}_{\mu\nu}$, which depends on the bulk solution. This term is reminiscent of the Weyl term in the codimension 1 brane worlds [14], which gives rise to the corrections to the Einstein equations on the brane. Roughly speaking, the brane world equation is obtained by taking the components of the full Einstein equations parallel to the brane, with the perpendicular components giving some information on the nature of the Weyl term. Depending on the symmetries present, in some cases (cosmology being the most physically interesting) we can completely determine the bulk metric, and hence Weyl corrections. For codimension 2, the perpendicular components of the bulk equations do lead to constraints that we discuss presently; however, these now no longer fix the bulk metric exactly, not even for the highly symmetric and special case of brane world cosmology with Einstein gravity in the bulk. Let us now investigate the consequences of (13), in particular, the consistency of the extra Weyl term, which arose as a result of allowing a discontinuity in the derivative of the parallel brane world metric.

A natural first check is to take the $\alpha \to 0$ limit to recover the Einstein case. Then Eq. (13) reduces to

$$2\pi(1-\beta)M_*^4\hat{g}_{\mu\nu} = \hat{T}_{\mu\nu}. \tag{14}$$

Although this looks like it is not possible to satisfy this matching condition unless the brane energy-momentum tensor is proportional to the induced metric, in fact, we have not yet determined whether β is a constant. A nonconstant β would correspond to a varying deficit angle and is not determined by the brane world equations alone. We must supplement the brane world equations with the bulk equations normal to the brane world, and since we wish to make as few assumptions as possible about the bulk in this Letter, we will simply look at the divergent $\mathcal{O}(1/r)$ terms in the Einstein equations near the brane, as these cannot be cancelled by any regular bulk $S_{\rm MN}$. These leading terms for the (μ, ν), (r, r), and (μ, r) components give

$$g_{\mu\nu}\frac{[L'']}{L} - \frac{L'}{2L}[\partial_r g_{\mu\nu} - g_{\mu\nu}g^{\rho\sigma}\partial_r g_{\rho\sigma}] = 0, \qquad \frac{L'}{2L}g^{\rho\sigma}\partial_r g_{\rho\sigma} = 0, \qquad \frac{\partial_\mu L'}{L} = 0, \tag{15}$$

where $[L'']$ stands here for the smooth part of the second derivative as we approach the brane. We now see directly that β must indeed be constant, and that $\partial_r^2 L|_{r=0^+} = 0$ and $\partial_r g_{\mu\nu}|_{r=0^+} = 0$. We now confirm the observation of Cline *et al.* [10]: that Einstein codimension 2 brane worlds must have an energy momentum proportional to their induced metric, and their gravitational effect is to produce a conical deficit in the bulk spacetime.

In Gauss-Bonnet gravity, however, the situation is not so simple, since all these equations get corrections proportional to α and one cannot rule out the existence of solutions with $\hat{W}_{\mu\nu} \neq 0$. The $\mathcal{O}(1/r)$ terms in the (μ, r) components of the EGB equations, for example, are

$$-g^{\nu\sigma}\frac{\partial_\sigma L'}{L}\left[g_{\mu\nu} + 4\alpha\left(R_{\mu\nu}(g) - \frac{1}{2}g_{\mu\nu}R(g)\right) - \alpha W_{\mu\nu}\right] + 2\alpha\frac{L'}{L}g^{\nu\sigma}[\partial_r g_{\mu\nu}R^\rho{}_{\sigma\rho r} - \partial_r g_{\nu\sigma}R^\rho{}_{\mu\rho r}] = 0, \tag{16}$$

with similar constraints from the $\mathcal{O}(1/r)$ terms of the (μ, ν) and (r, r) equations (though these are somewhat more complicated and not particularly illuminating). In this case we find that, in general, no simple restriction can be placed on the solution, and, in particular, the deficit angle β need no longer be constant.

However, it is important to note that some components of the Ricci curvature tensor (and scalar) are now divergent once we allow $\partial_r g_{\mu\nu}|_{0^+} \neq 0$. For example,

$$R_{\mu\nu} = \frac{1}{2}\frac{L'}{L}\partial_r g_{\mu\nu} + \cdots = \frac{\partial_r g_{\mu\nu}}{2r} + \mathcal{O}(1) \tag{17}$$

near the brane. In a realistic situation, we could argue that a brane would have finite width, which could act as a cutoff for the curvature; hence all our results would still be valid provided this cutoff is sufficiently large so that the curvature is still small compared to M_*^2, the six-dimensional Planck mass squared. In this smooth case, we *can* use the Gauss-Codazzi formalism and the θ independence of the metric to write

$$W_{\mu\nu} = K^\lambda_{i\mu}K_{i\nu\lambda} - K_iK_{i\mu\nu} + \frac{1}{2}g_{\mu\nu}[K_i^2 - K_{i\lambda\sigma}^2], \tag{18}$$

where $K_{i\mu\nu}$ are the two extrinsic curvatures ($i = 1, 2$) for each of the two normals. We therefore have the interpretation of $W_{\mu\nu}$ as a geometric correction to the Einstein tensor due to the embedding of the brane world in the bulk geometry. The interpretation is then that the Einstein equations acquire additional embedding terms, which unfortunately cannot be deduced from the brane world geometry alone.

The physical relevance of terms, which lead to divergent curvatures and hence tidal forces in the vicinity of the brane world, is, however, questionable. If M_* is of order the (inverse) brane width, or if we wish to have a truly infinitesimal brane, then we are forced to conclude that, for consistency, we cannot stop at the GB curvature corrections but must include all higher order curvature corrections, thus entering a nonperturbative regime of which we can say nothing. We are therefore forced to impose $\partial_r g_{\mu\nu} = 0$, and Eq. (16) tells us that the deficit angle β is again constant and the equation for the induced metric (13) remarkably takes the form of purely four-dimensional Einstein gravity:

$$\hat{G}_{\mu\nu} = \frac{1}{8\pi(1-\beta)\alpha M_*^4}\hat{T}_{\mu\nu} - \frac{1}{4\alpha}\hat{g}_{\mu\nu}. \tag{19}$$

We can read off our four-dimensional Planck mass as

$$M_{\rm Pl}^2 = 8\pi(1-\beta)\alpha M_*^4, \tag{20}$$

and we note the presence of an effective four-dimensional cosmological constant:

$$\Lambda_4 = T_0 - 2\pi(1-\beta)M_*^4, \tag{21}$$

where T_0 is the bare brane tension:

$$\hat{T}_{\mu\nu} = T_0\hat{g}_{\mu\nu} + \delta T_{\mu\nu}. \tag{22}$$

Of course the splitting of the energy-momentum tensor in this manner is potentially arbitrary, however, for a cosmological brane $\delta T_{\mu\nu} \to 0$ as $t \to \infty$, and we can simply posit that $\delta T_{\mu\nu} \to 0$ as either t or $|\mathbf{x}| \to \infty$ as being a necessary requirement of a brane world, thus rendering (22) unambiguous.

Interestingly, the Einstein relation between β and the brane tension, namely, $T_0 = 2\pi(1-\beta)M_*^4$, no longer holds for GB gravity—we can specify the conical deficit and the brane tension independently, the only caveat being that if the Einstein relation does not hold, then we have an effective cosmological constant on the brane.

To sum up, we have found the equations governing the induced metric on the brane for a codimension 2 brane world. We have shown that adding the Gauss-Bonnet term allows for a realistic gravity on an infinitesimally thin brane, which remarkably turns out to be precisely four-dimensional Einstein gravity *independent* of the precise bulk structure, the only bulk dependence appearing via the constant deficit angle Δ in the definition of the four-dimensional Planck mass $M_{\rm Pl}^2 = 4\alpha\Delta M_*^4$. Since Einstein gravity appears quite generically, our model provides a novel alternative realization of the infinite extra dimensions idea of Dvali *et al.* [15]. Indeed, we could modify our model by adding brane world Ricci terms (which can be motivated via finite width corrections to the brane effective action [12]), which would give the same form of the brane world gravity equations, and simply renormalize the four-dimensional Planck mass.

We also showed that it was possible to obtain a deviation from Einstein gravity via a nonzero $\hat{W}_{\mu\nu}$. In turn, this allows a variation of the bulk deficit angle and therefore the effective brane cosmological constant. In this case, one has to either perform a smooth regularization of the brane by taking some finite width vortex model or accept that the infinitesimally thin brane world has a nonperturbative regime in the neighborhood of the brane. Nevertheless, it seems to be a very appealing feature towards a possible solution of the cosmological constant problem. One could envisage a situation in which the system is in a nonperturbative phase in which the cosmological constant can vary and relax itself dynamically to a perturbative state in which the induced gravity on the brane is four-dimensional Einstein gravity and with a very small cosmological constant (an infinite flat supersymmetric bulk might, for instance, lead to this situation [16]). Because of the unbounded curvature near the brane when this situation is violated, it seems plausible that once the system reaches that configuration it would prefer to remain there.

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