Exact Decoherence to Pointer States in Free Open Quantum Systems is Universal

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In this Letter it is shown that exact decoherence to minimal uncertainty Gaussian pointer states is generic for free quantum particles coupled to a heat bath. More specifically, the Letter is concerned with damped free particles linearly coupled under product initial conditions to a heat bath at arbitrary temperature, with arbitrary coupling strength and spectral densities covering the Ohmic, sub-Ohmic, and supra-Ohmic regime. Then it is true that there exists a time t_c such that for times $t > t_c$ the state can always be exactly represented as a mixture (convex combination) of particular minimal uncertainty Gaussian states, regardless of and independent from the initial state. This exact "localization" is hence not a feature specific to high temperature and weak damping limit, but is a generic property of damped free particles.

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There is a long tradition of approaching the questions of how and to what extent classical properties of quantum systems emerge dynamically due to the unavoidable coupling to their environment. Essentially, any quantum system interacts to some extent with other external degrees of freedom, which in turn may be said to monitor certain properties of the quantum systems [1–10]. This yields decoherence, which results in a loss of purity of initially pure states of a distinguished quantum system coupled to an environment. Not all initial quantum states in such a dilation are yet equally "fragile" to this interaction: there is a small set of initial states that is often relatively robust with respect to this interaction. The term pointer states has been coined for such states, owing the name to models for quantum measurement where the pointer basis is essentially determined by the interaction of the apparatus with the external degrees of freedom [1].

For harmonic and free quantum systems linearly coupled to a heat bath consisting of harmonic systems, this general mechanism is very well understood indeed. For example, if one prepares a single mode in a pure state in order to let it very weakly interact with an environment in the Gibbs state corresponding to a very high temperature, which one is the state that produces the least entropy over one cycle of the oscillator? In retrospect it hardly comes as a surprise that this is a coherent state, a minimal uncertainty Gaussian state [6]. Most attention has probably been devoted to thoroughly understanding the dynamics of harmonic and free quantum systems in this limit of weak coupling and high temperatures [1–3,6]. In this limit, in particular, decoherence time scales have been identified [1]. Also exact quantum master equations, generators of dynamical maps, have been derived and scrutinized in great detail [9,11–14]. After all, the dynamics of open harmonic or free quantum systems cannot be described other than being well understood. What else is there to ask for?

A question that seems to have been overlooked so far is the following: To what extent is exact decoherence in free quantum systems to pointer states actually generic? This question is most appealing in case of the free damped quantum particle [12,14], where there is no equilibrium Gibbs state. More specifically, is it true that starting from an arbitrary quantum state, after a fixed finite time t_c (independent of the initial state), the state of the system is exactly indistinguishable from a mixture, a convex combination, of minimal uncertainty Gaussian states for all times $t > t_c$? In this sense, the free quantum system may be said to be in a situation that can operationally not be distinguished locally from the following situation: the particle is somewhere in a minimal uncertainty Gaussian state; one simply does not know where in phase space. That this is the case seems fairly plausible for the case of high temperatures and weak damping. A significant first step in this direction has indeed been achieved very recently by Diosi and Kiefer in Ref. [7], showing that this intuition is indeed correct for the approximate generator for the dynamical map in the limit of negligible friction and at high temperatures. Yet is this a generic feature of free quantum systems that are linearly coupled to an environment in a dilation, and true not only for specific regimes, but also for any coupling strength, any nonzero temperature, and Ohmic, sub-Ohmic, as well as supra-Ohmic damping? This is the question that will be addressed (and answered) in this Letter.

A free quantum system linearly coupled to a heat bath of oscillators will be investigated where the distinguished system is initially in an arbitrary (and potentially very "nonclassical") state, whereas the environment is prepared in the Gibbs state, which corresponds to an initial product state, such that the time evolution of the state of the free quantum system amounts to a completely positive dilation [15]. No assumptions will be made concerning the temperature of the environment and the strength of

the coupling; for the class of nonvanishing spectral densities, any C^∞ function $I:\mathbb{R}^+ \to \mathbb{R}^+$ could be allowed for with

$$\lim_{\omega\to 0} I(\omega)/\omega^p = \zeta > 0 \tag{1}$$

for some $p \in (0, 2)$. This will be referred to as Ohmic damping when $p = 1$, otherwise as sub-Ohmic (for $p < 1$) or as supra-Ohmic (for $p > 1$). This is an already solved problem in the sense that quantum master equations are known, and hence, the argument draws heavily from known results on generators of dynamical maps [11,17], and from earlier results on the long-time behavior in quantum Brownian motion [14]. The starting point is the equation of motion of the reduced density operators as derived in Ref. [11], in the integrated form as presented in the recent article in Ref. [17]. Later, ideas will be used very similar to the ones in Ref. [7].

The equation of motion of the free particle is for the subsequent purposes most conveniently expressed in phase space in terms of the Wigner function $W:\mathbb{R}^2 \times \mathbb{R}^+ \to \mathbb{R}$ [18], which is for each $t \in \mathbb{R}^+$ the Fourier transform of the characteristic function, dependent on $\xi = (\xi_1, \xi_2) \in \mathbb{R}^2$, where ξ_1 and ξ_2 correspond to position and momentum coordinates in phase space, respectively. As a partial differential equation, the Hu-Paz-Zhang equation [11] reads [19]

$$\begin{aligned}\partial_t W(\xi, t) = &-\xi_2 \partial_{\xi_1} W(\xi, t) + \Omega^2(t)\xi_1 \partial_{\xi_2} W(\xi, t)\\ &+ 2\Gamma(t)\partial_{\xi_2}(\xi_2 W(\xi, t)) + \Gamma(t)h(t)\partial_{\xi_2}^2 W(\xi, t)\\ &+ \Gamma(t)f(t)\partial_{\xi_1}\partial_{\xi_2} W(\xi, t),\end{aligned}$$

where the $\Gamma, f, h, \Omega:\mathbb{R}^+ \to \mathbb{R}$ are time-dependent coefficients for which explicit expressions are known. The formal solution of this partial differential equation can be found for all system parameters [17]. The solution of the differential equation with time-dependent coefficients as presented in Ref. [17] is given by

$$\begin{aligned}W(\xi, t) = &\int d^2\xi' \frac{1}{2\pi|M(t)|^{1/2}}\\ &\times e^{-(R(\xi,\xi',t)M(t)^{-1}R(\xi,\xi',t)^T)/2} W(\xi', 0) R(\xi, \xi', t)\\ = &(\xi_1 - \dot{G}(t)\xi_1' - G(t)\xi_2', \xi_2 - \ddot{G}(t)\xi_1' - \dot{G}(t)\xi_2'),\end{aligned}$$

where dots represent time derivatives. Here, $G:\mathbb{R} \to \mathbb{R}$ is the Green's function, which is $G(t) = 0$ for $t < 0$ and is for $t > 0$ the solution of the integral equation

$$\ddot{G}(t) + \int_0^t ds\,\gamma(t - s)\dot{G}(s) = 0,$$

$$\gamma(t) = \int_0^\infty d\omega \frac{I(\omega)}{\omega}\cos(\omega t),$$

with initial conditions $G(0) = 0$, $\dot{G}(0) = 1$, in terms of the so-called damping kernel. The 2×2 matrix

$$M(t) = \begin{bmatrix} A(t) & C(t) \\ C(t) & B(t) \end{bmatrix}$$

has coefficients that in Ref. [17] have been expressed in terms of correlation functions. On using the function $K:\mathbb{R}^+ \to \mathbb{R}$,

$$K(t) = \frac{1}{\pi}\int_0^\infty d\omega \mathrm{re}[\tilde{\gamma}(\omega + i0^+)]\,\omega\coth(\beta\omega)\cos(\omega t),$$

with $\tilde{\gamma} = \int_0^\infty dt\gamma(t)e^{izt}$, the coefficients $A(t)$, $B(t)$, and $C(t)$ can be expressed as

$$\begin{aligned}A(t) &= \int_0^t ds \int_0^t ds' G(t - s)G(t - s')K(s - s'),\\ B(t) &= \int_0^t ds \int_0^t ds' \dot{G}(t - s)\dot{G}(t - s')K(s - s'),\\ C(t) &= \int_0^t ds \int_0^t ds' G(t - s)\dot{G}(t - s')K(s - s'),\end{aligned}$$

as $G(0) = 0$. Equation (2), together with the subsequent specifications, forms the starting point of our analysis.

Equation (2), using the transformation rule for multiple integrals, can be written in the form of a product of a time-dependent determinant and a convolution with a Gaussian as

$$\begin{aligned}W(\xi, t) = &\int d^2\xi' \frac{1}{2\pi|M(t)|^{1/2}} e^{-((\xi-\xi')M(t)^{-1}(\xi-\xi')^T)/2}\\ &\times \frac{1}{|V(t)|} W(V(t)^{-1}\xi', 0),\\ V(t) = &\begin{bmatrix} \dot{G}(t) & G(t) \\ \ddot{G}(t) & \dot{G}(t) \end{bmatrix}.\end{aligned} \tag{2}$$

In general, the Green's function G can not be evaluated in a closed form, the case of Ohmic damping being an exception, where the spectral density is for small frequencies linear in the frequencies. The Laplace transform $\hat{G}$ of G is related to the Laplace transform $\hat{\gamma}$ of γ as $\hat{G}(z) = (z^2 + z\hat{\gamma}(z))^{-1}$. In order to specify the long-time behavior of the Green's functions, it is sufficient to know the power law for the spectral density for small frequencies only. Using Eq. (1), one arrives at $p \in (0, 2)$ at $\lim_{t\to\infty} G(t)/f(t) = 1$ (see also Ref. [14]), where $f(t) = \sin(\pi p/2)t^{p-1}/(\zeta\Gamma(p))$. From the asymptotic behavior of f as $t \to \infty$, it can be seen after a few steps that $\lim_{t\to\infty} A(t)/A'(t) = 1$, with

$$A'(t) = \int_{-\infty}^t ds \int_\infty^t ds' G(t - s)G(t - s')K(s - s').$$

This quantity, in turn, happens to be a quantity investigated in Ref. [14], where it has been shown that $\lim_{z\to 0}(\hat{A}'(z)\beta z)/(2\hat{G}(z)) = 1$, which yields $\lim_{t\to\infty} A(t)/A''(t) = 1$, with

$$A''(t) = \frac{2\sin(\pi p/2)}{\beta\zeta\Gamma(p + 1)} t^p. \tag{3}$$

In order to find the long-time behavior of the function C, we may use the fact that $C(t) = 2\dot{A}(t)$ for all $t \in [0, \infty)$, which holds since $G(0) = 0$, and apply l'Hospital's rule to arrive at $\lim_{t\to\infty} C(t)/C''(t) = 1$, with

$$C''(t) = \frac{2p\sin(\pi p/2)}{\beta\zeta\Gamma(p+1)}t^{p-1}. \quad (4)$$

To get the long term behavior of B, we can again start with $\lim_{t\to\infty}B(t)/B'(t) = 1$, where

$$B'(t) = \int_{-\infty}^{t} ds \int_{\infty}^{t} ds' \dot{G}(t-s)\dot{G}(t-s')K(s-s').$$

This, in turn, is nothing but the momentum uncertainty in the stationary setting, which is well defined even in this free case (compare also Ref. [17,20]),

$$\frac{1}{\pi}\int_0^\infty d\omega \mathrm{im}[\alpha(\omega + i0^+)]\omega^2\coth(\omega\beta) = B_\infty > 0, \quad (5)$$

with $\alpha[z] = -1/(z^2 + iz\tilde{\gamma}(z)]$. B_∞ is a (time-independent) positive real number. So we have determined the long-time behavior of the entries of the symmetric 2×2 matrix $M(t)$.

Subsequently, a pointer state is taken to be a minimal uncertainty Gaussian state with particular second moments that reflect a small uncertainty in the position canonical coordinate. The statements will be formulated in a language common in quantum optics and continuous-variable quantum information theory. The first moments are $(d_1, d_2) = (\langle X\rangle, \langle P\rangle)$; the second moments are collected in the covariance matrix

$$\Gamma = \begin{bmatrix} 2\langle O_1^2\rangle & \langle O_1O_2 + O_2O_1\rangle \\ \langle O_1O_2 + O_2O_1\rangle & 2\langle O_2^2\rangle \end{bmatrix},$$

where $O_1 = X - \langle X\rangle$ and $O_2 = P - \langle P\rangle$. The second moments for the pointer states are taken to be

$$\Gamma_\infty = \begin{bmatrix} B_\infty^{-1} & 0 \\ 0 & B_\infty \end{bmatrix}. \quad (6)$$

This is a covariance matrix of a minimal uncertainty state as $|\Gamma_\infty| = 1$. Note, that in the weak damping limit, B_∞ becomes approximately [21] $B_\infty = \beta^{-1} = T$ (note that $\hbar = 1$ and $k = 1$), so in the weak coupling and high temperature limit, the set of pointer states is a set of minimal uncertainty Gaussian states very narrow in position. The corresponding pure Gaussian state with first moments $(d_1, d_2) = (\xi_1, \xi_2)$ will be denoted as $\rho_\xi = |\psi_\xi\rangle\langle\psi_\xi|$. This set of minimal uncertainty Gaussian states, which becomes a set of states very narrow in position in the limit of weak coupling and high temperatures, will be regarded as the set of pointer states [22]. It is an overcomplete set of states satisfying $|\langle\psi_\xi|\psi_{\xi'}\rangle|^2 = e^{-(\xi-\xi')^T(\Gamma_\infty/2)(\xi-\xi')}$. The analogue of the standard s-ordered Wigner function of a state ρ may be defined as

$$W_s(\xi) = \frac{1}{\pi^2}\int d^2\xi' e^{s(\xi_1'^2+\xi_2'^2)/\sqrt{2}} e^{-2i\xi\sigma\xi'^T} \times \mathrm{tr}[e^{i\alpha\{X,P\}_+}e^{i(\xi_1X+\xi_2P)}e^{-i\alpha\{X,P\}_+}\rho],$$

$s\in[-1,1]$, where σ is the symplectic matrix embodying the canonical commutation relations, $\{.,.\}_+$ denotes the anticommutator, and $\alpha = -\log(B_\infty)/2$ is the squeezing parameter corresponding to the pointer states (taken with respect to the standard unit quantum oscillator). The state can then be represented as [23]

$$\rho = \int d^2\xi W_1(\xi)|\psi_\xi\rangle\langle\psi_\xi|, \quad (7)$$

whereas, in turn, $W_{-1}(\xi) = \langle\psi_\xi|\rho|\psi_\xi\rangle/\pi \geq 0$ for all $\xi\in\mathbb{R}^2$. Then, the s-ordered functions are related to each other via convolutions (compare, e.g., Ref. [24])

$$W_s(\xi) = \int d^2\xi' \frac{W_{s'}(\xi')}{2\pi}\frac{4}{s'-s}e^{-2(\xi-\xi')\Gamma_\infty^{-1}(\xi-\xi')^T/(s'-s)}, \quad (8)$$

for $s < s'$. We are now in the situation that we can argue similarly to Ref. [7]: the function W_0':$\mathbb{R}^2\times\mathbb{R}^+\to\mathbb{R}$, $W_0'(\xi,t) = W_0(V^{-1}(t)\xi, 0)/|V(t)|$, is a legitimate Wigner function, as can be read off the definition of the Wigner function. Then, Eqs. (2) and (8) imply that

$$W_1(\xi,t) = \int d^2\xi' \frac{W_0'(\xi',t)}{2\pi}|M(t) - \Gamma_\infty/4|^{-1/2} \times e^{-(\xi-\xi')(M(t)-\Gamma_\infty/4)^{-1}(\xi-\xi')^T/2}.$$

However, since

$$\int d^2\xi' \frac{W_0'(\xi',t)}{2\pi}4e^{-2(\xi-\xi')\Gamma_\infty(\xi-\xi')^T} \geq 0$$

for all $\xi\in\mathbb{R}^2$, then $W_1(\xi,t)\geq 0$ for all $\xi\in\mathbb{R}^2$ if

$$M(t) - \Gamma_\infty/4 \geq \Gamma_\infty/4. \quad (9)$$

In turn, given the time dependence of the coefficients of $M(t)$ demonstrated in Eqs. (3)–(5), there exists a finite $t_c > 0$ such that (9) is valid for all $t > t_c$. This time t_c, in turn, is the time from which W_1 is strictly positive, and the state can certainly exactly be represented as a mixture of pointer states with second moments as in Eq. (6).

This is a generic result for arbitrary nonzero temperatures, arbitrary coupling strengths, and all the spectral densities as in Eq. (1). For specific choices for the spectral density, bounds for the time t_c can be found from which the state can be represented as a mixture of pointer states. For Ohmic damping, in particular, the Green's function is given by $G(t) = (1 - e^{-\zeta t})/\zeta$, i.e., $\hat{\gamma}(z) = \zeta > 0$ [14,17]. The behavior becomes particularly transparent in the high temperature case. We then simply obtain $\lim_{T\to\infty}A(t)/T = 2(t - G(t))/\zeta$, $\lim_{T\to\infty}C(t)/T = 2(1-\dot{G}(t))/\zeta$, and $\lim_{T\to\infty}B(t)/T = 1 - e^{-2\zeta t}$. Figure 1 depicts $T_c = \lim_{T\to\infty}t_c$, where t_c is the smallest time for which (9) is satisfied for strictly Ohmic damping.

To conclude, it has been shown that if one couples a free particle linearly to a heat bath prepared in the Gibbs state of some temperature, then, under very general conditions and without approximations, the state of the system becomes after some finite time exactly indistinguishable from an exact mixture of particular minimal uncertainty Gaussian pointer states. In this sense, it can be said that exact decoherence to these localized pointer states is generic and not only a feature of a limit that can be regarded as being classical. Locally, hence, we arrive at the situation as if we had merely classical ignorance about the position of the particle. Needless to say, care is required in the interpretation of the result, and one should not be tempted by a realistic interpretation in terms of

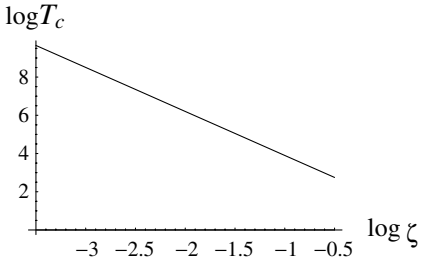

FIG. 1. This figure shows $\log T_c = \lim_{T\to\infty} \log t_c$, where for a given temperature the number, $t_c > 0$ is the smallest number such that (9) is satisfied for all $t > t_c$ for the case of strictly Ohmic damping, as a function of $\log\zeta$. The stronger the damping, the faster is the localization process.

classical alternatives. In turn, the total state of both the system and its environment is very different in structure and is typically a highly correlated and often, but not necessarily [5], entangled state. It is the hope that this Letter can contribute to the debate on the dynamical appearance of classical properties in quantum theory. This debate is potentially becoming more timely than ever with the availability of novel experiments on decoherence [10], let it be with microwave cavities, ion traps, or nano-electromechanical systems.

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$$\sigma = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix},$$

with σ being the symplectic matrix such that $\xi\sigma\xi'^T$ is the symplectic scalar product between $\xi, \xi' \in \mathbb{R}^2$.

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