Supersymmetric Patterns in the Pseudospin, Spin, and Coulomb Limits of the Dirac Equation with Scalar and Vector Potentials

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We show that the Dirac equation in $(3 + 1)$ dimensions gives rise to supersymmetric patterns when the scalar and vector potentials are (i) Coulombic with arbitrary strengths or (ii) when their sum or difference is a constant, leading to relativistic pseudospin and spin symmetries. The conserved quantities and the common intertwining relation responsible for such patterns are discussed.

The Dirac equation for spin $1/2$ particles plays a central role in the relativistic description of atoms, nuclei, and hadrons. In atoms the relevant potentials felt by the electron (or muon in muonic atoms) are Coulombic vector potentials. Relativistic mean fields in nuclei generated by meson exchanges [1] and quark confinement in hadrons [2] necessitate a mixture of Lorentz vector and scalar potentials. Recently, symmetries of Dirac Hamiltonians with such Lorentz structure have been shown to be relevant for explaining the observed degeneracies of certain shell-model orbitals in nuclei ("pseudospin doublets") [3] and the absence of quark spin-orbit splitting ("spin doublets") [4], as observed in heavy-light quark mesons. The goal of the current Letter is to show that the degeneracy patterns and relations between wave functions implied by such relativistic symmetries resemble the patterns found in supersymmetric schemes. The underlying mechanism responsible for such properties will be examined. The feasibility of such a proposal gains support from the fact that Dirac Hamiltonians with selected external fields are known to be supersymmetric [5], e.g., for a vector Coulomb potential [6].

Supersymmetric quantum mechanics (SUSYQM), initially proposed as a model for supersymmetry (SUSY) breaking in field theory [7], has by now developed into a field in its own right, with applications in diverse areas of physics [8]. The essential ingredients of SUSYQM are the supersymmetric Hamiltonian $\mathcal{H} = \begin{pmatrix} H_1 & 0 \\ 0 & H_2 \end{pmatrix} = \begin{pmatrix} L^\dagger L & 0 \\ 0 & L L^\dagger \end{pmatrix}$ and charges $Q_- = \begin{pmatrix} 0 & 0 \\ L & 0 \end{pmatrix}$, $Q_+ = \begin{pmatrix} 0 & L^\dagger \\ 0 & 0 \end{pmatrix}$ which generate the supersymmetric algebra $[\mathcal{H}, Q_\pm] = \{Q_\pm, Q_\pm\} = 0$, $\{Q_-, Q_+\} = \mathcal{H}$. The partner Hamiltonians H_1 and H_2 satisfy an intertwining relation, $L H_1 = H_2 L$, where in one dimension the transformation operator $L = \frac{d}{dx} + W(x)$ is a first-order Darboux transformation expressed in terms of a superpotential $W(x)$. The intertwining relation ensures that, if Ψ_1 is an eigenstate of H_1, then also $\Psi_2 = L\Psi_1$ is an eigenstate of H_2 with the same energy, unless $L\Psi_1$ vanishes or produces an unphysical state (e.g., non-normalizable). Consequently, as shown in Fig. 1(a), the SUSY partner Hamiltonians H_1 and H_2 are isospectral in the sense that their spectra consist of pairwise degenerate levels with a possible nondegenerate

single state in one sector (when the supersymmetry is exact). The wave functions of the degenerate levels are simply related in terms of L and $L^\dagger$. Such characteristic features define a supersymmetric pattern. The intertwining relation ensures such properties for any pair of Hamiltonians not necessarily factorizable. We will continue to use the term "supersymmetric pattern" also in such circumstances. In what follows, we focus the discussion on supersymmetric patterns obtained in selected Dirac Hamiltonians.

The Dirac Hamiltonian, H, for a fermion of mass M moving in external scalar V_S and vector V_V potentials is given by $H = \hat{\boldsymbol{\alpha}} \cdot \boldsymbol{p} + \hat{\boldsymbol{\beta}}(M + V_S) + V_V$, where $\hat{\boldsymbol{\alpha}}$, $\hat{\boldsymbol{\beta}}$ are the usual Dirac matrices [9], and we have set $\hbar = c = 1$.
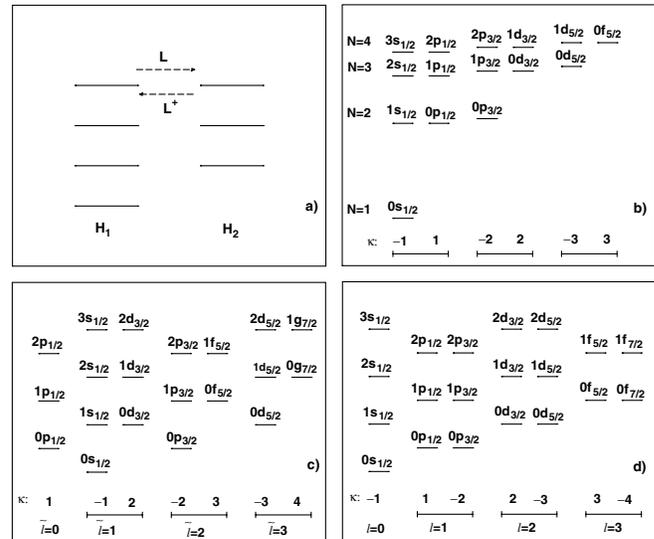
FIG. 1. Schematic qualitative supersymmetric patterns in (a) SUSYQM and in the (b) Coulomb, (c) pseudospin, and (d) spin limits of the Dirac Hamiltonian. In (a), H_1 and H_2 have identical spectra with an additional level for H_1 when SUSY is exact. Spectroscopic notation $n\ell j$ in (b)–(d) refers to quantum numbers of the upper component, and κ, N, and $\tilde{\ell}$ are defined in the text. In (b) the radial nodes n are related to n_r by $n_r = n$ ($n_r = n + 1$) for $\kappa < 0$ ($\kappa > 0$), and only the $E_{n_r,\kappa}^{(+)}$ branch is shown.

When the potentials are spherically symmetric: $V_S = V_S(r)$, $V_V = V_V(r)$, the operator $\hat{K} = -\hat{\beta}(\boldsymbol{\sigma} \cdot \ell + 1)$ (with $\boldsymbol{\sigma}$ the Pauli matrices and $\ell = -ir \times \boldsymbol{\nabla}$), commutes with H and its nonzero integer eigenvalues $\kappa = \pm(j + 1/2)$ are used to label the Dirac wave functions $\Psi_{\kappa, m} = r^{-1}(G_\kappa[Y_\ell \chi]_m^{(j)}, iF_\kappa[Y_{\ell'} \chi]_m^{(j)})$. Here $G_\kappa(r)$ and $F_\kappa(r)$ are the radial wave functions of the upper and lower components, respectively; Y_ℓ and χ are the spherical harmonic and spin function which are coupled to angular momentum j with projection m. The labels $\kappa = -(j + 1/2) < 0$ and $\ell' = \ell + 1$ hold for aligned spin $j = \ell + 1/2$ ($s_{1/2}, p_{3/2}$, etc.), while $\kappa = (j + 1/2) > 0$ and $\ell' = \ell - 1$ hold for unaligned spin $j = \ell - 1/2$ ($p_{1/2}, d_{3/2}$, etc.). Denoting the pair of radial wave functions by

$$\Phi_\kappa = \begin{pmatrix} G_\kappa \\ F_\kappa \end{pmatrix}, \tag{1}$$

the radial Dirac equations can be cast in Hamiltonian form, $H_\kappa \Phi_\kappa = E \Phi_\kappa$, with

$$H_\kappa = \begin{pmatrix} M + \Delta & -\frac{d}{dr} + \frac{\kappa}{r} \\ \frac{d}{dr} + \frac{\kappa}{r} & -(M + \Sigma) \end{pmatrix}, \tag{2a}$$

$$\Delta(r) = V_S + V_V, \qquad \Sigma(r) = V_S - V_V. \tag{2b}$$

We now look for Dirac Hamiltonians H_{κ_1} and H_{κ_2} which satisfy an intertwining relation of the form

$$LH_{\kappa_1} = H_{\kappa_2}L. \tag{3}$$

Following [10], we consider a matrical Darboux transformation operator,

$$L = A(r)\frac{d}{dr} + B(r), \tag{4}$$

where A and B are 2×2 matrices with r-dependent entries $A_{ij}(r)$, $B_{ij}(r)$. Relations (3) and (4) should be regarded as a system of equations for the unknown operator L and the so-far unspecified potentials in H_κ (2). The matrices $A(r)$ and $B(r)$ are found to be

$$A_{11} = A_{22} = a, \qquad A_{12} = -A_{21} = b, \tag{5a}$$

$$B_{11} = -b(M + \Delta) - \frac{1}{2r}a\omega_-(\omega_+ + 1) + \frac{1}{2}c_2, \tag{5b}$$

$$B_{22} = b(M + \Sigma) - \frac{1}{2r}a\omega_-(\omega_+ - 1) + \frac{1}{2}c_2, \tag{5c}$$

$$B_{12} = aV_V - \frac{1}{2r}b\omega_+(\omega_- + 1) + \frac{1}{2}c_1, \tag{5d}$$

$$B_{21} = -aV_V + \frac{1}{2r}b\omega_+(\omega_- - 1) - \frac{1}{2}c_1, \tag{5e}$$

where $\omega_+ = (\kappa_1 + \kappa_2)$, $\omega_- = (\kappa_2 - \kappa_1)$ and a, b, c_1, c_2 are constants. In addition, the following relations have to be obeyed:

$$2a\left[V_S' + V_V\frac{\omega_+}{r}\right] - b\frac{\omega_-(\omega_+ + 1)(\omega_+ - 1)}{r^2} + c_1\frac{\omega_+}{r} = 0, \tag{6a}$$

$$a\left[-4V_V(M + V_S) + \frac{\omega_+(\omega_- + 1)(\omega_- - 1)}{r^2}\right] + 2b\frac{\omega_-}{r}[\omega_+(M + V_S) + V_V] - c_2\frac{\omega_-}{r} - 2c_1(M + V_S) = 0, \tag{6b}$$

where V_S' denotes differentiation with respect to r. In the usual application of SUSYQM, one starts from a solvable Hamiltonian H_1 and uses the intertwining relation to obtain a new solvable Hamiltonian H_2. In the present case, we employ a different strategy, namely, insist that both partner Hamiltonians H_{κ_1} and H_{κ_2} be of the form prescribed in Eq. (2) with the *same* potentials, and look for solutions of Eq. (6) such that the potentials are independent of κ. We find that there are six such solutions characterized by $\omega_+, \omega_- = 0, \pm 1$. The solution with $\omega_- = 0$ is trivial ($\kappa_1 = \kappa_2$), $L = -bH_\kappa + \frac{1}{2}c_2 I$, where I is the 2×2 unit matrix. The solutions with $\omega_- = \pm 1$ lead to constant potentials $V_S = S_0$ and $V_V = V_0$. The physically interesting solutions require $\omega_+ = 0, 1, -1$ and lead to the Coulomb, pseudospin, and spin limits, respectively.

The Coulomb limit ($\kappa_1 + \kappa_2 = 0$).—The solutions of Eq. (6) with $\omega_+ = \kappa_1 + \kappa_2 = 0$ fix the potentials to be of Coulomb type:

$$V_S = \frac{\alpha_S}{r} + S_0, \qquad V_V = \frac{\alpha_V}{r} + V_0, \tag{7}$$

with arbitrary strengths, α_S, α_V. The constants S_0 and V_0 amount to constant shifts in the mass and Hamiltonian, respectively, and henceforth will be omitted. In terms

of the quantities $\eta_1 = (\alpha_S M + \alpha_V E)/\lambda$, $\eta_2 = (\alpha_S E + \alpha_V M)/\lambda$, $\lambda = \sqrt{M^2 - E^2}$, and $\gamma = \sqrt{\kappa^2 + \alpha_S^2 - \alpha_V^2}$, the quantization condition reads $\gamma + \eta_1 = -n_r$ ($n_r = 0, 1, 2, \dots$), and leads to the eigenvalues [9] $E_{n_r,\kappa}^{(\pm)}/M = [-\alpha_S\alpha_V \pm \xi\sqrt{\xi^2 - \alpha_S^2 + \alpha_V^2}]/(\alpha_V^2 + \xi^2)$, where $\xi = (n_r + \gamma)$, and the κ dependence enters through the factor γ. The spectrum consists of two branches denoted by superscripts $(+)$ and $(-)$. The corresponding eigenfunctions are

$$\Phi_{n_r,\kappa} \propto \begin{pmatrix} -\sqrt{M + E}[(\kappa + \eta_2)F_1 + n_r F_2] \\ \sqrt{M - E}[(\kappa + \eta_2)F_1 - n_r F_2] \end{pmatrix}\rho^\gamma e^{-\rho/2}, \tag{8}$$

where $E = E_{n_r,\kappa}^{(\pm)}$ and $F_1 = F(-n_r, 2\gamma + 1, \rho)$, $F_2 = F(-n_r + 1, 2\gamma + 1, \rho)$ are confluent hypergeometric functions in the variable $\rho = 2\lambda r$. The states and energies in each branch are labeled by (n_r, κ). It is also possible to express the results in terms of the principal quantum number N defined as $N = n_r + |\kappa|$, ($N = 1, 2, \dots$). For $n_r \geq 1$, the eigenvalues in each branch are twofold degenerate with respect to the sign of κ, i.e., $E_{n_r,\kappa}^{(+)} = E_{n_r,-\kappa}^{(+)}$ and $E_{n_r,\kappa}^{(-)} = E_{n_r,-\kappa}^{(-)}$. For $n_r = 0$, there is only *one* acceptable

state for each κ, which has $\kappa < 0$ for the (+) branch and $\kappa > 0$ for the (−) branch. Equivalently, for a fixed principal quantum number N, the allowed values of κ are $\kappa = \pm 1, \pm 2, \ldots, \pm(N-1), -N$ for the (+) branch and $\kappa = \pm 1, \pm 2, \ldots, \pm(N-1), +N$ for the (−) branch of the spectrum.

Focusing on the set of states with $\kappa_1 = -\kappa_2 \equiv \kappa$, the levels are separated according to the value of $|\kappa| = j + 1/2$. For fixed κ, $E_{n_r,\kappa}^{(+)}$ is an increasing function of n_r and, as shown in Fig. 1(b), for each value of j we have a characteristic supersymmetric pattern. There are two towers of energy levels, one for $-|\kappa|$ (with $n_r = 0, 1, 2, \ldots$) and one for $+|\kappa|$ (with $n_r = 1, 2, \ldots$). The two towers are identical, except that the $E_{n_r=0,-|\kappa|}^{(+)}$ level at the bottom of the $-|\kappa|$ tower is nondegenerate. Similar patterns of pairwise degenerate levels with $\pm\kappa$ appear also in the (−) branch of the spectrum. However, since for fixed κ, $E_{n_r,\kappa}^{(-)}$ is a decreasing function of n_r, the nondegenerate $E_{n_r=0,|\kappa|}^{(-)}$ level is now at the top of the $+|\kappa|$ tower, resulting in an inverted supersymmetric pattern. From Eqs. (5) and (6), we find the transformation operator to be

$$L = a \begin{pmatrix} \frac{d}{dr} + \frac{\epsilon_+}{r} + \frac{M\alpha_+}{\kappa_1} & -\frac{\alpha_S}{\kappa_1}\frac{d}{dr} + \frac{\alpha_V}{r} \\ \frac{\alpha_S}{\kappa_1}\frac{d}{dr} - \frac{\alpha_V}{r} & \frac{d}{dr} - \frac{\epsilon_-}{r} - \frac{M\alpha_-}{\kappa_1} \end{pmatrix}, \quad (9)$$

where $\epsilon_\pm = \kappa_1 + \alpha_S \alpha_\pm / \kappa_1$ and $\alpha_\pm = (\alpha_S \pm \alpha_V)$. The operator L connects degenerate states with $(n_r \geq 1, \pm\kappa)$, and annihilates the nondegenerate states with $n_r = 0$:

$$L\Phi_{n_r,\kappa_1} = C\Phi_{n_r,\kappa_2} \qquad (\kappa_1 = -\kappa_2). \quad (10)$$

Here $C = (a\lambda/\kappa_1)\sqrt{n_r(\gamma - \eta_1)}$ and $\Phi_{n_r,\kappa}$ are given in Eq. (8). Constructing supersymmetric charges $Q_\pm$ and Hamiltonian $\mathcal{H}$ from L and H_{κ_1}, H_{κ_2} in the manner described at the beginning of the Letter, ensures that $[\mathcal{H}, Q_\pm] = \{Q_\pm, Q_\pm\} = 0$, but now $\{Q_-, Q_+\} \propto [\mathcal{H} - E_{n_r=0,\kappa}^{(+)}][\mathcal{H} - E_{n_r=0,\kappa}^{(-)}]$, with $\kappa = \kappa_1 = -\kappa_2$. These relations represent a quadratic deformation of the conventional supersymmetric algebra [10], which arises because both the Dirac Hamiltonian and the transformation operator L are of first order.

The explicit solvability and observed degeneracies of the relativistic Coulomb problem are related to the existence of an additional conserved Hermitian operator:

$$B = -i\hat{K}\gamma_5(H - \hat{\beta}M) + \frac{\boldsymbol{\sigma}\cdot\boldsymbol{r}}{r}(\alpha_V M + \alpha_S H), \quad (11)$$

which commutes with the full Dirac scalar and vector Coulomb Hamiltonian H but anticommutes with $\hat{K}$. This operator is a generalization of the Johnson-Lippmann operator applicable for $\alpha_S = 0$ [11].

The pseudospin limit ($\kappa_1 + \kappa_2 = 1$).—The solutions of Eq. (6) with $\omega_+ = \kappa_1 + \kappa_2 = 1$ require that the sum of scalar and vector potentials is a constant:

$$\Delta(r) = V_S(r) + V_V(r) = \Delta_0. \quad (12)$$

Under such a condition, the Dirac Hamiltonian is invariant under an SU(2) algebra, whose generators are [12,13]

$$\hat{\tilde{\boldsymbol{S}}}_\mu = \begin{pmatrix} \hat{\tilde{s}}_\mu & 0 \\ 0 & \hat{s}_\mu \end{pmatrix}. \quad (13)$$

Here $\hat{s}_\mu = \sigma_\mu/2$ are the usual spin operators, $\hat{\tilde{s}}_\mu = U_p \hat{s}_\mu U_p$ and $U_p = \frac{\boldsymbol{\sigma}\cdot\boldsymbol{p}}{p}$. This relativistic symmetry has been used [3] to explain the occurrence of pseudospin doublets in nuclei [14]. The latter refer to the empirical observation of quasidegenerate pairs of certain normal-parity shell-model orbitals with nonrelativistic single-nucleon radial, orbital, and total angular momentum quantum numbers: $(n, \ell, j = \ell + 1/2)$ and $(n-1, \ell + 2, j = \ell + 3/2)$. The doublet structure is expressed in terms of a "pseudo" orbital angular momentum, $\tilde{\ell} = \ell + 1$, and pseudo spin, $\tilde{s} = 1/2$, which are coupled to $j = \tilde{\ell} \pm \tilde{s}$. For example, $[ns_{1/2}, (n-1)d_{3/2}]$ will have $\tilde{\ell} = 1$, etc., Such doublets play a central role in explaining features of nuclei [15], including superdeformation and identical bands. In a relativistic description of nuclei [1], these nonrelativistic wave functions are identified with the upper components of Dirac wave functions, $\Psi_{\kappa_1<0,m}$ and $\Psi_{\kappa_2>0,m}$ with $\kappa_1 + \kappa_2 = 1$, which are eigenstates of a Dirac Hamiltonian with scalar and vector mean field potentials, approximately satisfying condition (12). The corresponding lower components have n nodes [16] and orbital angular momentum equal to $\tilde{\ell}$ [3]. In the pseudospin limit, these two Dirac states form a degenerate doublet, and their radial components satisfy $F_{\kappa_1} = F_{\kappa_2}$, and $(dG_{\kappa_1}/dr) + (\kappa_1/r)G_{\kappa_1} = (dG_{\kappa_2}/dr) + (\kappa_2/r)G_{\kappa_2}$. These relations have been tested in numerous realistic mean field calculations in a variety of nuclei, and were found to be obeyed to a good approximation, especially for doublets near the Fermi surface [17,18]. For potentials with asymptotic behavior as encountered in nuclei, the Dirac eigenstates, for which both the upper (G_κ) and lower (F_κ) components have no nodes, can occur only for $\kappa < 0$ and, hence, do not have a degenerate partner eigenstate (with $\kappa > 0$) [16]. These nodeless Dirac states correspond to the shell-model states with $(n = 0, \ell, j = \ell + 1/2)$. For heavy nuclei, such states with large j, i.e., $0g_{9/2}, 0h_{11/2}, 0i_{13/2}$, are the "intruder" abnormal-parity states which, indeed, empirically are found not to be part of a doublet [15]. Altogether, as shown in Fig. 1(c), the ensemble of Dirac states with $\kappa_1 + \kappa_2 = 1$ exhibits a supersymmetric pattern of twin towers with pairwise degenerate pseudospin doublets sharing a common $\tilde{\ell}$, and an additional nondegenerate nodeless state at the bottom of the $\kappa_1 < 0$ tower. An exception to this rule is the tower with $\kappa_2 = 1$ ($p_{1/2}$ states with $\tilde{\ell} = 0$), which is on its own, because states with $\kappa_1 = 0$ do not exist. From Eqs. (5) and (6), we find the transformation operator to be

$$L = b \begin{pmatrix} 0 & \frac{d}{dr} - \frac{\kappa_2}{r} \\ -\frac{d}{dr} - \frac{\kappa_1}{r} & (2M + \Sigma + \Delta_0) \end{pmatrix}. \quad (14)$$

L connects the two doublet states as in Eq. (10) but with $\kappa_1 + \kappa_2 = 1$ and $C = b(M + \Delta_0 - E)$. In this case, $\{Q_-, Q_+\} = b^2[\mathcal{H} - (M + \Delta_0)][\mathcal{H} - (M + \Delta_0)]$ is proportional to a polynomial of $\mathcal{H}$, again indicating a deformation of the conventional SUSY algebra. In real nuclei, the relativistic pseudospin symmetry is slightly broken, implying $\Delta(r) \neq \Delta_0$ in Eq. (12). Taking H_κ as in Eq. (2) and L as in Eq. (14) but with $\Delta_0 \rightarrow \Delta(r)$, we find that $LH_{\kappa_1} - H_{\kappa_2}L = ib\Delta'\sigma_2$. Furthermore, $\{Q_-, Q_+\}$ has the same formal form as before, but the appearance of $\Delta(r)$ instead of Δ_0 implies that the anticommutator is no longer just a polynomial of $\mathcal{H}$.

The spin limit ($\kappa_1 + \kappa_2 = -1$).—The solutions of Eq. (6) with $\omega_+ = \kappa_1 + \kappa_2 = -1$ require that the difference of the scalar and vector potentials is a constant

$$\Sigma(r) = V_S(r) - V_V(r) = \Sigma_0. \tag{15}$$

Under such a condition, the Dirac Hamiltonian is invariant under another SU(2) algebra, whose generators are obtained from Eq. (13) by interchanging $\hat{s}_\mu$ and $\hat{\tilde{s}}_\mu$ [12]:

$$\hat{S}_\mu = \begin{pmatrix} \hat{s}_\mu & 0 \\ 0 & \hat{\tilde{s}}_\mu \end{pmatrix}. \tag{16}$$

This relativistic symmetry gives rise to degenerate doublets of Dirac states $\Psi_{\kappa_1 < 0, m}$ and $\Psi_{\kappa_2 > 0, m}$ with $\kappa_1 + \kappa_2 = -1$, whose upper components have quantum numbers $(n, \ell, j = \ell + 1/2)$ and $(n, \ell, j = \ell - 1/2)$. Such spin doublets were argued to be relevant for degeneracies observed in heavy-light quark mesons [4] and possibly for the antinucleon spectrum in a nucleus [19]. In the spin limit, the corresponding radial components satisfy $G_{\kappa_1} = G_{\kappa_2}$ and $(dF_{\kappa_1}/dr) - (\kappa_1/r)F_{\kappa_1} = (dF_{\kappa_2}/dr) - (\kappa_2/r)F_{\kappa_2}$. As shown in Fig. 1(d), the spectrum consists of towers of states with $\kappa_1 + \kappa_2 = -1$, forming pairwise degenerate spin doublets. In this case, none of the towers have a single nondegenerate state and, hence, the spectrum corresponds to that of a broken SUSY [8]. The tower with $\kappa_1 = -1$ ($s_{1/2}$ states) is on its own, since states with $\kappa_2 = 0$ do not exist. The transformation operator, found from Eqs. (5) and (6),

$$L = -b\begin{pmatrix} (2M + \Sigma_0 + \Delta) & -\frac{d}{dr} + \frac{\kappa_1}{r} \\ \frac{d}{dr} + \frac{\kappa_2}{r} & 0 \end{pmatrix}, \tag{17}$$

connects the two doublet states as in Eq. (10), but with $\kappa_1 + \kappa_2 = -1$ and $C = -b(E + M + \Sigma_0)$. The nilpotent charges $Q_\pm$ commute with the supersymmetric Hamiltonian $\mathcal{H}$ and exhibit a deformation of the conventional SUSY algebra, $\{Q_-, Q_+\} = b^2[\mathcal{H} + (M + \Sigma_0)] \times [\mathcal{H} + (M + \Sigma_0)]$. When $\Sigma(r) \neq \Sigma_0$ in Eq. (15), we have $LH_{\kappa_1} - H_{\kappa_2}L = -ib\Sigma'\sigma_2$, where H_κ is given in Eq. (2) and L, as well as $\{Q_-, Q_+\}$, have the same form as before but with $\Sigma_0 \rightarrow \Sigma(r)$.

In summary, a common intertwining relation was shown to provide the basis for a unified treatment of three separate limits at which a Dirac Hamiltonian, with scalar and vector potentials, exhibits supersymmetric patterns.

In the Coulomb limit, the potentials are $1/r$ but their strengths are otherwise arbitrary. In the pseudospin or spin limits, there are no restrictions on the r dependence of the potentials but there is a constraint on their sum or difference. The characteristic degeneracies reflect the presence of additional conserved operators, the generalized Johnson-Lippmann operator given in Eq. (11), and the previously introduced relativistic pseudospin and spin generators [12,13]. It is gratifying to note that the indicated supersymmetric patterns are manifested empirically, to a good approximation, in physical dynamical systems. While previous studies have focused on individual doublets in nuclei and hadrons, it is the grouping of several doublets (and intruder levels in nuclei), as suggested in the present work, which highlights the fingerprints of supersymmetry present in these dynamical systems.

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