Regularization of the Linearized Gravitational Self-Force for Branes

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We discuss the linearized, gravitational self-interaction of a brane of arbitrary codimension in a spacetime of arbitrary dimension. We find that in the codimension two case the gravitational self-force is exactly zero for a Nambu-Goto equation of state, generalizing a previous result for a string in four dimensions. For the case of a 3-brane, this picks out the case of a six-dimensional brane-world model as having special properties that we discuss. In particular, we see that bare tension on the brane has no effect locally, suppressing the cosmological constant problem.

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The divergent self-force of a charged point particle, such as the electron, coupled to electromagnetism has been understood for many years. Its resolution via the inclusion of an ultraviolet (UV) cutoff, due to the finite radius of the particle, leads to a renormalization of the particle's mass and a suppression of the pole singularity at short distances (see, for example, [1]).

This problem is not unique, and, in fact, similar problems exist for any distributional source coupled to any kind of field in any spacetime dimension. An interesting case is that of a Nambu-Goto string coupled to linearized gravity. It has been shown [2–4] that the self-force, regularized in the UV by the core width of the string, ϵ, and in the infrared (IR) by the interstring separation, Δ, is exactly zero due to the fact that the induced linearized metric perturbation is orthogonal to the string world sheet. This result can be shown to be true at all orders in perturbation theory, in the case of a static string [5].

A similar result can be deduced when the string in four dimensions is coupled to an axion field, represented by a 2-form, and a dilaton, as well as linearized gravity [6]. For a special choice of couplings, which was predicted [7] in the context of $\mathcal{N} = 1$, $D = 10$ supergravity, one can show that the combined self-interaction is zero; the dilaton contribution is negative, which cancels the positive contribution from the axion field.

It should be noted that the UV regularization of the self-field is not necessary in the codimension one case, the hypersurface, where the behavior at the brane can be dealt with using junction conditions. The case of gravity can be dealt with exactly, at all orders, using the conditions often attributed to Israel [8] (although see Ref. [9]). Similar lines of argument lead to the junction conditions at a surface in Maxwell's theory of electromagnetism [1].

The extension of these ideas to higher dimensions has become more relevant recently with the interest that has arisen in brane-world models. In these models, the matter of the standard model of particle physics is confined to a four-dimensional subspace, or *brane,* of a higher-dimensional spacetime, often called the *bulk.* Two types of model have received particular attention: six-dimensional models with flat, compact extra dimensions, such as the Arkani-Hamed–Dimopoulos–Dvali model [10], and five-dimensional models with warped extra dimensions, such as the Randall-Sundrum (RS) models [11]. These ideas were originally motivated by the notion of D-branes in M theory, and the desire to alleviate the weak hierarchy problem of UV quantum field theory (QFT). However, much subsequent work has focused on their gravitational properties.

Both models can be extended to higher dimensions. As we have discussed, the five-dimensional case has no UV divergence, so it does not need to be regularized. When the extra dimensions are compact, the volume of the extra dimensions gives an effective IR cutoff scale. In the warped case, the curvature length scale of the bulk spacetime fulfills a similar role. In more general cases it is clear that some physical phenomena must provide either a UV cutoff (usually the thickness of the brane) or one in the IR (usually the distance between branes or the background curvature length scale). In the codimension two case, one requires both since the self-field is proportional to $1/r$ and the divergence of the self-energy is logarithmic.

One intriguing aspect of brane worlds with two extra dimensions [12] is that the bare tension of the brane, which represents vacuum energy, does not appear to gravitate from the point of view of an observer on the brane, its effects being felt only in the bulk as a modification to the conical deficit angle. This can be thought of as a *self-tuning* model, suppressing the cosmological constant problem since the large variations in the vacuum energy expected due to, for example, cosmological phase transitions would not be experienced gravitationally by observers on the brane.

As we have described, the study of self-interactions finds applications in a wide range of research areas, from cosmic defects to superstring and M theory. In this Letter we discuss systematically the regularization of the gravitational self-force for extended objects with any codimension more than one, albeit at linearized order. Our results are also relevant to the codimension one case, but, as we have already noted, they are not completely necessary there. We find that, in the case of a Nambu-Goto brane, the self-force takes a simple form and can be interpreted as a renormalization of the tension. In the codimension two case, this renormalization is exactly zero, extending the result for cosmic strings [2–4] to hyperstrings in arbitrary spacetime dimension. Our analysis allows for a general configuration of the brane and for background curvature on a scale greater than the effective width of the brane. It is, therefore, an extension of the self-tuning cosmological constant idea, in that previous work [12] has considered only symmetric, exact solutions in specific background spacetimes. We consider the analogue of Refs. [6], which includes the effect of a dilaton and an antisymmetric form field, in a more detailed forthcoming paper.

We consider a p-brane, with a $(p+1)$-dimensional world sheet, in an n-dimensional spacetime. The position of the brane is given in terms of the spacetime coordinates x^μ by $x^\mu = X^\mu\{\sigma^a\}$, where σ^a are internal world sheet coordinates. The induced metric on the brane is then given by $\gamma_{ab} = g_{\mu\nu}\partial_a X^\mu \partial_b X^\nu$, and the background energy-momentum tensor, $\hat{T}^{\mu\nu}$, due to that supported on the world sheet, $\overline{T}^{\mu\nu}$, is

$$\hat{T}^{\mu\nu}\{x\} = \frac{1}{\sqrt{-g}} \int \overline{T}^{\mu\nu} \delta^{(n)}\{x - X\{\sigma^a\}\} \sqrt{-\gamma}\, d^{p+1}\sigma. \quad (1)$$

The first fundamental tensor of the brane and its orthogonal complement can then be defined as $\eta^{\mu\nu} = \gamma^{ab}\partial_a X^\mu \partial_b X^\nu$ and $\perp_{\mu\nu} = g_{\mu\nu} - \eta_{\mu\nu}$, respectively. These act as the projection operators tangential and orthogonal to the world sheet. The second fundamental tensor of the world sheet and the extrinsic curvature vector are defined as $K_{\mu\nu}{}^\rho = \eta_\mu{}^\alpha \eta_{\nu\beta} \nabla_\alpha \eta^{\beta\rho}$ and $K^\rho = g^{\mu\nu} K_{\mu\nu}{}^\rho$. This formulation in terms of background tensorial quantities has the advantage of avoiding the complications of the internal indices, σ^a. In the case of a codimension one brane, $\perp_{\mu\nu} = n_\mu n_\nu$, $K_{\mu\nu} = K_{\mu\nu}{}^\rho n_\rho$, and $K^\rho = K n^\rho$ where n_ρ is the unit normal covector to the brane, and $K_{\mu\nu}$, K are the more familiar extrinsic curvature pseudotensor and scalar, respectively. Note that we use the sign conventions of [13], whereas some authors define the extrinsic curvature with the opposite sign.

We perform our regularization calculation in a flat background spacetime, but it is also valid in the case where the background is curved as long as the associated curvature scale is larger than the brane thickness. In the case of an anti–de Sitter (AdS) background this requires that the AdS length scale $l \gg \epsilon$. Moreover, we also demand that the curvature scale of the brane be much larger than the brane thickness, i.e., $\sqrt{K^\rho K_\rho} \gg \epsilon$. This condition does not hold, for example, at a cusp in the brane world sheet.

We consider a perturbation of the metric $g_{\mu\nu} \to g_{\mu\nu} + h_{\mu\nu}$ with $|h_{\mu\nu}| \ll 1$, which, in an n-dimensional Minkowski spacetime, satisfies the linearized Einstein equation

$$\Box h_{\mu\nu} = -2(n-2)\Omega^{[n-2]} G\left(\hat{T}_{\mu\nu} - \frac{\hat{T} g_{\mu\nu}}{n-2}\right), \quad (2)$$

where $\Box = \nabla_\rho \nabla^\rho$ is the wave operator defined by the unperturbed metric, $G = M^{2-n}$ is the gravitational coupling constant appropriate to the background spacetime, and $\Omega^{[n]}$ is the area of a unit n sphere.

In order to perform the regularization, we make the split of the metric perturbation $h_{\mu\nu} = \hat{h}_{\mu\nu} + \tilde{h}_{\mu\nu}$, where $\hat{h}_{\mu\nu}$ is the singular contribution from the string and $\tilde{h}_{\mu\nu}$ is the (finite) remainder due to radiation backreaction and external effects. Defining the standard Green's function $\mathcal{G}\{x, X(\sigma^a)\}$ for the wave operator in Minkowski space, we find that

$$h_{\mu\nu}\{x\} = \beta \int \left(\overline{T}_{\mu\nu} - \frac{\overline{T} g_{\mu\nu}}{n-2}\right) \mathcal{G}\{x, X\{\sigma^a\}\} \sqrt{-\gamma}\, d^{p+1}\sigma, \quad (3)$$

where $\beta = -2(n-2)\Omega^{[n-2]}G/\Omega^{[n-1]}$. Using the standard form for the Green's function, this solution can be regularized, both in the UV and in the IR, to give

$$\hat{h}_{\mu\nu} = 2G\left(\overline{T}_{\mu\nu} - \frac{\overline{T} g_{\mu\nu}}{n-2}\right) F_{\{\Delta,\epsilon\}}, \quad (4)$$

where we have defined a regularization factor

$$F_{\{\Delta,\epsilon\}} = \frac{\Omega^{[n-2]}\Omega^{[p]}}{\Omega^{[n-1]}} \int_\epsilon^\Delta s^{p+2-n} ds, \quad (5)$$

to describe the dependence on the IR and UV cutoffs. At its simplest level this represents a hard cutoff in source density for $s < \epsilon$ and $s > \Delta$, but the cutoffs could easily be thought of as effective, representing the envelope of a solution. The factor $F_{\{\Delta,\epsilon\}}$ encapsulates all of the dependence on the internal structure of the string and the effect of spacetime compactification or curvature on $\hat{h}_{\mu\nu}$. The regularization can be justified by a more rigorous calculation as described in [14].

The effective UV cutoff scale, ϵ, is governed by the internal structure of the brane. Except in the codimension one case, the infinitely thin limit leads to a divergence, meaning that the profile of the brane is always important. The profile of the brane removes the divergence associated with an infinitely thin source, thus generating an effective thickness, which we assume is the same at every point on the world sheet. For the domain wall case, such as the RS models [11], there is no UV divergence and the infinitely thin limit can be used.

The IR cutoff scale, Δ, is necessary when considering branes of codimension one or two (domain walls and hyperstrings). The IR divergence can be removed in several ways, each of which generates an effective value of Δ. One such way is the usual Kaluza-Klein approach where one compactifies the extra dimensions on a circle or torus, the radius or volume of the internal space giving the effective cutoff scale. Another possibility is to consider an AdS bulk, as in the RS model, where the different form of the Green's function means that the corresponding integral does not have an IR divergence. The AdS length scale, l, then gives an effective cutoff, $\Delta \sim l$, allowing us to use the solution (4). Another possibility is to consider a network of branes where the interbrane separation gives an IR cutoff scale, as is usually assumed to be the case for $p = 1$, $n = 4$, i.e., cosmic strings.

One can determine the force acting on the brane by considering the variation of the brane component to the matter action

$$S = \int \overline{\mathcal{L}} \sqrt{-\gamma}\, d^{p+1}\sigma. \tag{6}$$

Under the perturbation $g_{\mu\nu} \to g_{\mu\nu} + h_{\mu\nu}$, the first order change in the Lagrangian is given by

$$\overline{\mathcal{L}}_{\text{eff}} = \overline{\mathcal{L}} + \tfrac{1}{4} h_{\rho\sigma} \overline{T}^{\rho\sigma}, \tag{7}$$

an extra factor of $1/2$ being the adjustment required to make sure the contributions are counted only once. Varying the action [2] shows us that the force on the brane is given by

$$\bar{f}^{\mu} = \tfrac{1}{2}\overline{T}^{\nu\rho}\nabla^{\mu} h_{\nu\rho} - \nabla_{\nu}(\overline{T}^{\nu\rho} h^{\mu}_{\rho} + \overline{C}^{\mu\nu\rho\sigma} h_{\rho\sigma}), \tag{8}$$

where the *hyper-Cauchy tensor* is defined by

$$\overline{C}^{\mu\nu\rho\sigma} = \frac{1}{\sqrt{-\gamma}} \frac{\delta}{\delta g_{\mu\nu}} (\sqrt{-\gamma}\, \overline{T}^{\rho\sigma}), \tag{9}$$

which is a relativistic version of the Cauchy elasticity tensor.

In order the evaluate the regularized force from (8), we need to compute the regularized version of the gradient $\nabla_{\rho} h_{\mu\nu}$. We do this using the formula derived in Ref. [14] for scalar field ϕ,

$$\widehat{\nabla_{\rho}\phi} = \eta^{\sigma}_{\rho} \nabla_{\sigma} \hat{\phi} + \tfrac{1}{2} K_{\rho} \hat{\phi}, \tag{10}$$

which are applied to each of the components of $h_{\mu\nu}$ since we are considering linearized interactions. This formula applies when $p > 0$ and when the codimension is greater than one. The factor of $1/2$ in the second term should be replaced by $(p - 1)/2p$ in the codimension one case. The first term in this formula is just the derivative tangent to the brane, which is all one would have in the case of a straight brane. The effect of the curvature is in the second term, which accounts for the change in orientation of the planes normal to the brane when it is curved within the bulk.

For a Nambu-Goto–type brane, the Lagrangian, energy-momentum tensor, and hyper-Cauchy tensor are given by $\overline{\mathcal{L}} = -\lambda = -m^{p+1}$, $\overline{T}^{\mu\nu} = -\lambda \eta^{\mu\nu}$ and

$$C^{\mu\nu\rho\sigma} = \lambda(\eta^{\mu(\rho}\eta^{\sigma)\nu} - \tfrac{1}{2}\eta^{\mu\nu}\eta^{\rho\sigma}), \tag{11}$$

where m is a fixed mass scale and λ is the tension of the brane. The singular part of the metric perturbation is given by

$$\hat{h}_{\mu\nu} = 2\lambda G F_{\{\Delta,\epsilon\}} \left(\frac{p + 3 - n}{n - 2} \eta_{\mu\nu} + \frac{p + 1}{n - 2} \perp_{\mu\nu} \right), \tag{12}$$

which, in the codimension two case where $p = n - 3$, is purely orthogonal to the world sheet.

If one regularizes the force (8), using the relation (10) for the regularized gradient and the solution (12), one can deduce that the linearized, gravitational self-force is given by

$$\hat{f}^{\mu} = \frac{(p + 1)(p + 3 - n)}{2(n - 2)} \lambda^2 F_{\{\Delta,\epsilon\}} K^{\mu}, \tag{13}$$

for a brane of codimension greater than one. In the codimension one case, there is an additional factor of $(p + 1)/p$. The force is in the direction of the extrinsic curvature vector, K^{ρ}, which is normal to the brane world sheet. This can be interpreted as a renormalization of the tension of the brane

$$\frac{\lambda_{\text{eff}}}{\lambda} = 1 - \frac{(p + 1)(p + 3 - n)}{2(n - 2)} \lambda G F_{\{\Delta,\epsilon\}}. \tag{14}$$

This renormalization represents a correction to the Lagrangian $\overline{\mathcal{L}}$ of the matter supported on the brane, providing a term that looks like an effective cosmological constant on the brane. It is obvious that this force, and hence the renormalization, vanishes when $p = n - 3$, generalizing the result previously derived for cosmic strings in four dimensions [3,4].

One can see this in much simpler terms, if one considers the action renormalization, that is, if one substitutes (12) into (7); Eq. (14) can be rederived very easily by using the fact that $\eta_{\mu\nu}\eta^{\mu\nu} = p + 1$, and one can then see directly that if $\hat{h}_{\mu\nu} \propto \perp_{\mu\nu}$, as is the case when $p = n - 3$, then not only is the self-force zero, but so is the action renormalization.

For a brane world, $p = 3$, and this special case requires $n = 6$. These phenomena have been pointed out recently by several authors [12] who have studied explicit solutions for specific formulations of six-dimensional brane worlds. Here, we have generalized this result to arbitrary, nonstatic configurations at linearized order. One can see that for a general surface energy-momentum tensor $\overline{T}^{\mu\nu}$, the action renormalization is given by

$$\mathcal{L}_{\text{eff}} = \mathcal{L} + \frac{1}{2} G F_{\{\Delta,\epsilon\}} \left(\overline{T}^{\mu\nu} \overline{T}_{\mu\nu} - \frac{1}{n - 2} \overline{T}^2 \right). \tag{15}$$

Some of these authors have suggested that this could have implications for the observed gravitational effects of vacuum energy in these models since the vacuum energy can be thought of as being the origin of the bare tension of the brane world. Effective QFTs predict that the energy scale associated with the vacuum could be as large as the Planck mass, $M_{\rm pl}$, making it discrepant by a factor of the order 10^{120} with the upper bounds from observation. If $n = 6$ and $p = 3$ then our calculation has a simple, but elegant resolution to this problem; the bare tension gravitates only in the direction orthogonal to the brane world, that is, $\hat{h}_{\mu\nu} \propto \perp_{\mu\nu}$, the gravitational self-force is zero, and the renormalized action has no constant contribution. Hence, the effective cosmological constant as experienced by observers on the brane would be zero, providing a self-tuning mechanism for the cosmological constant. This establishes a link between the self-tuning phenomena at work here and the self-force.

We note that there are some obvious problems associated with this self-tuning mechanism, not least the fact that there is evidence to suggest that the cosmological constant is nonzero [15]. Moreover, observations of the angular power spectrum of anisotropies in the cosmic microwave background [16] suggest that inflation is their ultimate origin. Both of these require cosmic acceleration, albeit from a scalar field in the case of inflation, which could not be due to any kind of brane-based effect in this scenario since *all* vacuum energy can do is modify the bulk solution with no observable effect on the brane.

One resolution of this would be to generate accelerated expansion from bulk effects. A bulk cosmological constant will gravitate and give an effective cosmological constant as seen by a brane-based observer. Similarly, accelerated expansion could be driven by a bulk scalar field. Of course, one still has a cosmological constant problem in the bulk: how to determine what mechanism fixes this to be small. Furthermore, reheating after bulk-induced inflation could be problematic if the inflaton is not formed from brane-based matter.

To summarize: the main result of this Letter is the generalization of the nondivergence of the self-force of a Nambu-Goto cosmic string to a corresponding hyperstring in arbitrary dimensions under the assumption that the solution is regularized by some physical phenomena in the UV and the IR. In fact, we have derived a general formula for the linearized gravitational self-force in arbitrary codimension and the corresponding renormalization of the bare tension. We have pointed out the links with recent work on attempts to self-tune the cosmological constant in 6D brane-world models. An analysis using a method similar to this but considering the gravitational interaction of observable matter supported on the brane, in the usual brane-world limit where it is small compared to the bare tension, would give a much firmer foundation to these ideas. There have been some studies of 6D brane worlds with matter [17], and these point to some potential problems: in those models considered it is possible to have only very specific matter distributions supported on the brane. We hope to investigate this in future work.

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[1] J. D. Jackson, *Classical Electrodynamics* (John Wiley and Sons Inc., New York, 1962).
[2] R. A. Battye and B. Carter, Phys. Lett. B **357**, 29 (1995).
[3] B. Carter and R. A. Battye, Phys. Lett. B **430**, 49 (1998).
[4] A. Buonanno and T. Damour, Phys. Lett. B **432**, 51 (1998).
[5] G.W. Gibbons, Classical Quantum Gravity **16**, 1471 (1999).
[6] E. Copeland, D. Haws, and M. Hindmarsh, Phys. Rev. D **42**, 726 (1990); A. Dabholkar, G.W. Gibbons, J. A. Harvey, and F. Ruiz-Ruiz, Nucl. Phys. **B340**, 33 (1990); B. Carter, Int. J. Theor. Phys. **38**, 2779 (1999).
[7] A. Dabholkar and J. A. Harvey, Phys. Rev. Lett. **63**, 478 (1989).
[8] W. Israel, Nuovo Cimento B **44**, 1 (1966); **48**, 463(E) (1966).
[9] K. Lanczos, Phys. Z. **23**, 539 (1922); K. Lanczos, Ann. Phys. (Leipzig) **74**, 518 (1924); N. Sen, Ann. Phys. (Leipzig) **73**, 365 (1924); G. Darmois, Mémorial des Sciences Mathématique **XXV** (1927); C.W. Misner and D. H. Sharp, Phys. Rev. **136**, B571 (1964).
[10] N. Arkani-Hamed, S. Dimopoulos, and G. Dvali, Phys. Lett. B **429**, 263 (1998); I. Antoniadis, N. Arkani-Hamed, S. Dimopoulos, and G. Dvali, Phys. Lett. B **436**, 257 (1998).
[11] L. Randall and R. Sundrum, Phys. Rev. Lett. **83**, 3370 (1999); **83**, 4690 (1999).
[12] A. G. Cohen and D. B. Kaplan, Phys. Lett. B **470**, 52 (1999); T. Gherghetta and M. Shaposhnikov, Phys. Rev. Lett. **85**, 240 (2000); P. Kanti, R. Madden, and K. A. Olive, Phys. Rev. D **64**, 044021 (2001); S. M. Carroll and M. M. Guica, hep-th/0302067; Y. Aghababaie, C. P. Burgess, S. L. Parameswaran, and F. Quevedo, hep-th/0304256; I. Navarro, Classical Quantum Gravity **20**, 3603 (2003); H.-P. Nilles, A. Papazoglou, and G. Tasinato, hep-th/0309042.
[13] C.W. Misner, K. S. Thorne, and J. A. Wheeler, *Gravitation* (W. H. Freeman and Co., New York, 1973).
[14] B. Carter, R. A. Battye, and J.-P. Uzan, Commun. Math. Phys. **235**, 289 (2003).
[15] S. Perlmutter *et al.*, Astrophys. J. **517**, 565 (1999).
[16] D. N. Spergel *et al.*, Astrophys. J. Suppl. Ser. **148**, 175 (2003); H.V. Peiris *et al.*, Astrophys. J. Suppl. Ser. **148**, 213 (2003).
[17] J. M. Cline, J. Descheneau, M. Giovannini, and J. Vinet, J. High Energy Phys. **0306** (2003) 048; P. Bostock, R. Gregory, I. Navarro, and J. Santiago, hep-th/0311074.