Fixed Points of Quantum Gravity

Daniel F. Litim*

Theory Division, CERN, CH-1211 Geneva 23
and School of Physics and Astronomy, University of Southampton, Southampton SO17 1BJ, United Kingdom
(Received 15 December 2003; published 19 May 2004)

Euclidean quantum gravity is studied with renormalization group methods. Analytical results for a nontrivial ultraviolet fixed point are found for arbitrary dimensions and gauge fixing parameters in the Einstein-Hilbert truncation. Implications for quantum gravity in four dimensions are discussed.

Classical general relativity is acknowledged as the theory of gravitational interactions for distances sufficiently large compared to the Planck length. At smaller length scales, quantum effects are expected to become important. The quantization of general relativity, however, still poses problems. It has long been known that quantum gravity is perturbatively nonrenormalizable, meaning that an infinite number of parameters have to be fixed to renormalize standard perturbation theory. It has been suggested that Einstein gravity may be nonperturbatively renormalizable, a scenario known as asymptotic safety [1]. This requires the existence of a nontrivial ultraviolet fixed point with at most a finite number of unstable directions. Then it would suffice to adjust a finite number of parameters, ideally taken from experiment, to make the theory asymptotically safe. Nonperturbative renormalizability has already been established for a number of field theories [2].

The search for fixed points in quantum field theory calls for a renormalization group study. A particularly useful approach is given by the Exact Renormalization Group, based on the integrating out of momentum modes from a path integral representation of the theory [3]. The strength of the method is its flexibility when it comes to approximations. General optimization procedures are available [4], increasing the domain of validity and the convergence of the flow. Consequently, the reliability of results based on optimized flows is enhanced [5].

Explicit flow equations for Euclidean quantum gravity have been constructed by Reuter [6], using background field techniques [6–8]. Diffeomorphism invariance under local coordinate transformations is controlled by modified Ward identities, similar to those known for non-Abelian gauge theories [9]. In general, methods originally developed for gauge theories [10], with minor modifications, can now be applied to quantum gravity.

So far most studies have been concerned with the Einstein-Hilbert truncation based on the operators $\sqrt{g}$ and $\sqrt{g}R(g)$ in the effective action, where g is the determinant of the metric tensor $g_{\mu\nu}$ and $R(g)$ the Ricci scalar. In four dimensions, the high energy behavior of quantum gravity is dominated by a nontrivial ultraviolet fixed point [7,11,12], which is stable under the inclusion of

$R^2(g)$ interactions [8] or noninteracting matter fields [13]. Further indications for the existence of a fixed point are based on dimensionally reduced theories [14] and on numerical studies within simplicial gravity [15]. For phenomenological applications, see [16].

In this Letter, we study fixed points of quantum gravity in the approach put forward in [6], amended by an adequate optimization [4,5]. The main new result is the existence of a nontrivial ultraviolet fixed point in the Einstein-Hilbert truncation in dimensions higher than 4, a region which previously has not been accessible. Analytical results for the flow and its fixed points are given for arbitrary dimension. The optimization ensures the maximal reliability of the result in the present truncation, thereby strengthening earlier findings in four dimensions. Implications of these results are discussed.

The Exact Renormalization Group is based on a momentum cutoff for the propagating degrees of freedom and describes the change of the scale-dependent effective action Γ_k under an infinitesimal change of the cutoff scale k. Thereby it interpolates between a microscopic action in the ultraviolet and the full quantum effective action in the infrared, where the cutoff is removed. In its modern formulation, the renormalization group flow of Γ_k with the logarithmic scale parameter $t = \ln k$ is given by

$$\partial_t \Gamma_k = \tfrac{1}{2}\text{Tr}(\Gamma_k^{(2)} + R)^{-1}\partial_t R. \tag{1}$$

The trace stands for a sum over indices and a loop integration, and R (not to be confused with the Ricci scalar) is an appropriately defined momentum cutoff at the momentum scale $q^2 \approx k^2$. For quantum gravity, we consider the flow (1) for $\Gamma_k[g_{\mu\nu}]$ in the Einstein-Hilbert truncation, retaining the volume element and the Ricci scalar as independent operators. Apart from a classical gauge fixing, the effective action is given by

$$\Gamma_k = \frac{1}{16\pi G_k} \int d^d x \sqrt{g}[-R(g) + 2\bar{\lambda}_k]. \tag{2}$$

In (2) we introduced the gravitational coupling constant G_k and the cosmological constant $\bar{\lambda}_k$. The ansatz (2) differs from the Einstein-Hilbert action in d Euclidean dimensions in that the couplings have turned into scale-dependent functions. Denoting $\bar{G}$ and $\bar{\lambda}$ the

unrenormalized Newtonian coupling and cosmological constant at some reference scale $k = \Lambda$, and $Z_{N,k}$ the wave function renormalization factor for the Newtonian coupling, we introduce dimensionless renormalized couplings as

$$g_k = k^{d-2} G_k \equiv k^{d-2} Z_{N,k}^{-1} \bar{G}, \qquad \lambda_k = k^{-2} \bar{\lambda}_k. \qquad (3)$$

Their flows follow from (1) by an appropriate projection onto the operators in (2). A scaling solution of the flow (1) in the truncation (2) corresponds to fixed points for the couplings (3).

Explicit momentum cutoffs have been provided for the fluctuations in the metric field in the Feynman gauge [6] and for its component fields in a traceless transverse decomposition in a harmonic background field gauge with the gauge fixing parameter α [7]. In either case the tensor structure of the regulator is fixed, while the scalar part is left free. Here, we employ the optimized cutoff $R_{opt} = (k^2 - q^2) \theta(k^2 - q^2)$ for the scalar part [4,5]. In the setup (1)–(3), we have computed the flow equation for arbitrary α and arbitrary dimension. To simplify the notation, a factor $1/\alpha$ is absorbed into the definition (3). Below we present explicit formulas only for the limit $\alpha \to \infty$ where the results take their simplest form. The general case is discussed elsewhere. The β functions are

$$\beta_\lambda \equiv \partial_t \lambda = \frac{P_1}{P_2 + 4(d+2)g},$$
$$\beta_g \equiv \partial_t g = (d - 2 + \eta)g = \frac{(d-2)g P_2}{P_2 + 4(d+2)g}, \qquad (4)$$

with polynomials $P_{1,2}(\lambda, g; d)$,

$$P_1 = -16\lambda^3 + 4\lambda^2(4 - 10dg - 3d^2 g + d^3 g)$$
$$+ 4\lambda(10dg + d^2 g - d^3 g - 1)$$
$$+ d(2 + d)(d - 16g + 8dg - 3)g,$$
$$P_2 = 8(\lambda^2 - \lambda - dg) + 2.$$

A numerical factor $c_d = \Gamma(\frac{d}{2} + 2)(4\pi)^{d/2-1}$ originating mainly from the momentum trace in (1) is scaled into the gravitational coupling $g \to g/c_d$, unless indicated otherwise. The graviton anomalous dimension is given by

$$\eta = \frac{(d-2)(d+2)g}{(d-2)g - 2(\lambda - \frac{1}{2})^2}. \qquad (5)$$

It vanishes for vanishing gravitational coupling, in two and minus two dimensions, or for diverging λ. On a nontrivial fixed point the vanishing of β_g implies $\eta_* = 2 - d$ and reflects the fact that the gravitational coupling is dimensionless in $2d$. This behavior leads to modifications of the graviton propagator at large momenta, e.g., [7]. The flows (4) are finite except on the boundary $g_{bound}(\lambda) \equiv (2\lambda - 1)^2/(2d - 4)$ derived from $1/\eta = 0$. It signals a breakdown of the truncation (2). Some trajectories terminate at $g \approx g_{bound}$. The boundary is irrelevant as soon as the couplings enter the domain of canonical scaling: the limit of large $|\lambda|$ implies that $\beta_\lambda = -2\lambda$ and

$\beta_g = (d - 2)g$ modulo subleading corrections. This entails $\eta = 0$. Then the flow is trivially solved by $g_k = g_\Lambda(k/\Lambda)^{d-2}$ and $\lambda_k = \lambda_\Lambda(k/\Lambda)^{-2}$, the canonical scaling of the couplings as implied by (3). Here, Λ denotes the momentum scale where the canonical scaling regime is reached. In the infrared limit $g/g_{bound}(\lambda) \sim (k/\Lambda)^{d+2}$ becomes increasingly small for any dimension.

In the remaining part, we discuss the fixed points of (4) and their implications. A first understanding is achieved in the limit of vanishing cosmological constant, where

$$\beta_g = \frac{(1 - 4dg)(d - 2)g}{1 + 2(2 - d)g}. \qquad (6)$$

The flow (6) displays two fixed points: the Gaussian one at $g = 0$ and a non-Gaussian one at $g_* = 1/(4d)$. In the vicinity of the nontrivial fixed point for large k, the gravitational coupling behaves as $G_k \approx g_*/k^{d-2}$. This behavior is similar to asymptotic freedom in Yang-Mills theories. The anomalous dimension of the graviton (5) remains finite and ≤ 0 for all g between the infrared and the ultraviolet fixed point, because $g_k \leq g_* < g_{bound}(0)$ for all $d > -2$ and $g_\Lambda \leq g_*$. At the fixed point, universal observables are the eigenvalues of the stability matrix at criticality, i.e., the critical exponents. The universal eigenvalue θ is given by $\partial \beta_g / \partial g|_* = -\theta$. We find

$$\theta_G = 2 - d, \qquad \theta_{NG} = 2d\frac{d-2}{d+2} \qquad (7)$$

for the Gaussian (G) and the non-Gaussian (NG) fixed points, respectively. The eigenvalues at criticality have opposite signs, the Gaussian one being infrared attractive and the non-Gaussian one being ultraviolet attractive. They are degenerate in two dimensions. Away from the fixed points, the flow (6) can be solved analytically for arbitrary scales k. With the initial condition g_Λ at $k = \Lambda$, the solution g_k for any k is

$$\left(\frac{g_k}{g_\Lambda}\right)^{-1/\theta_G}\left(\frac{g_* - g_k}{g_* - g_\Lambda}\right)^{-1/\theta_{NG}} = \frac{k}{\Lambda}. \qquad (8)$$

Figure 1 shows the crossover from the infrared to the ultraviolet fixed point in the analytic solution (8) in four
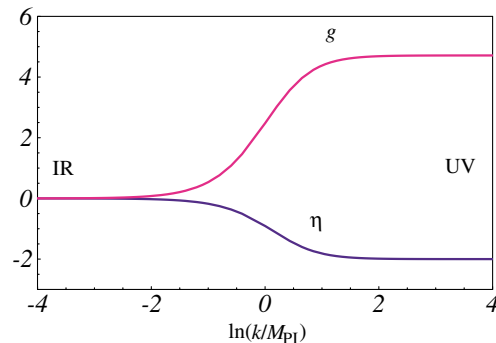

FIG. 1 (color online). Running of couplings according to (6) and (8).

dimensions (and with c_4 in g reinserted). The corresponding crossover momentum scale is associated with the Planck mass, $M_{Pl} = (G)^{-1/2}$. More generally, (8) is a solution at $\lambda = 0$ for an arbitrary gauge fixing parameter and a regulator. In these cases, the exponents θ and the fixed point g_* turn into functions of the latter. It is reassuring that the eigenvalues display only a mild dependence on these parameters.

Now we proceed to the nontrivial fixed points implied by the simultaneous vanishing of β_g and β_λ, given in (3). A first consequence of $P_2 = 0$ is $g_* = (\lambda_* - \frac{1}{2})^2/d$, which, when inserted into $P_1/g_* = 0$, reduces the fixed point condition to a quadratic equation with two real solutions $(\lambda_*, g_*) \neq 0$ as long as $d \geq d_+$, where $d_\pm = \frac{1}{2}(1 \pm \sqrt{17})$. The two branches of fixed points are characterized by λ_* being larger or smaller than $\frac{1}{2}$. The branch with $\lambda_* \gtrsim \frac{1}{2}$ displays an unphysical singularity at four dimensions and is therefore discarded. Hence,

$$\lambda_* = \frac{d^2 - d - 4 - \sqrt{2d(d^2 - d - 4)}}{2(d-4)(d+1)},$$

$$g_* = \frac{2\Gamma(\frac{d}{2} + 2)(4\pi)^{d/2-1}}{d^2 - d - 4}\lambda_*^2. \tag{9}$$

In (9), we have reinserted the numerical factor c_d. The solutions (9) are continuous and well defined for all $d \geq d_+ \approx 2.56$ and become complex for lower dimensions. The critical exponents associated with (9) are derived from the stability matrix at criticality. In the most interesting case $d = 4$, the two eigenvalues are a complex conjugate pair $\theta_\pm \equiv \theta' \pm i\theta'' = (5 \pm i\sqrt{167})/3$, or

$$\theta' = 1.667, \qquad \theta'' = 4.308. \tag{10}$$

The eigenvalues remain complex for all dimensions $2.56 < d < 21.4$. In $d = 4$, and for the general gauge fixing parameter, the eigenvalues vary between $\theta' \approx$ 1.5–2 and $\theta'' \approx$ 2.5–4.3. The range of variation serves as an indicator for the self-consistency of (2). The result (10) and the variation with α agree well with earlier findings based on other regulators [7,11]. In the approximation (6), which does not admit complex eigenvalues, the ultraviolet eigenvalue reads $\theta_{NG} = 8/3$ in $4d$. Both θ_{NG} and θ' agree reasonably well with the critical eigenvalue $\theta \approx 3$ detected within a numerical study of $4d$ simplicial gravity with fixed cosmological constant [15]. In view of the conceptual differences between the numerical analysis of [15] and the present approach, the precise relationship between these findings requires further clarification. Still, the qualitative agreement is very encouraging.

Next, we discuss the main characteristics of the phase portrait defined through (4). Finiteness of the flow implies that the line $1/\eta = 0$ cannot be crossed. Slowness of the flow implies that the line $\eta = 0$ cannot be crossed either: in its vicinity, the running of g is $\beta_g = (d-2)g + \mathcal{O}(g^2)$, and the gravitational coupling approaches $g = 0$ without ever reaching (or crossing) it for any scale k.

Thus, disconnected regions of renormalization group trajectories are characterized by whether g is larger or smaller g_{bound} and by the sign of g. Since η changes sign only across the lines $\eta = 0$ or $1/\eta = 0$, we conclude that the graviton anomalous dimension has the same sign along any trajectory. In the physical domain which includes the ultraviolet and the infrared fixed points, the gravitational coupling is positive and the anomalous dimension negative. In turn, the cosmological constant may change sign on trajectories emanating from the ultraviolet fixed point. Some trajectories terminate at the boundary $g_{bound}(\lambda)$, linked to the present truncation (cf. Fig. 2). The two fixed points are connected by a separatrix. In Fig. 2, it has been given explicitly in four dimensions (with the factor c_4 in g reinstalled). Integrating the flow starting in the vicinity of the ultraviolet fixed point and fine-tuning the initial condition leads to the trajectory which runs into the Gaussian fixed point. The rotation of the separatrix about the ultraviolet fixed point reflects the complex nature of the eigenvalues (10). At $k \approx M_{Pl}$, the flow displays a crossover from ultraviolet dominated running to infrared dominated running. For the running couplings and η this behavior is displayed in Fig. 3. The nonvanishing cosmological constant modifies the flow mainly in the crossover region rather than in the ultraviolet. A similar behavior is expected for operators beyond the truncation (2). This is supported by the stability of the fixed point under $R^2(g)$ corrections [8], and by the weak dependence on the gauge fixing parameter. In the infrared limit, the separatrix leads to a vanishing cosmological constant $\bar{\lambda}_k = \lambda_k k^2 \to 0$. Therefore, it is interpreted as a phase transition boundary between cosmologies with positive or negative cosmological constant at large distances. This picture agrees very well with numerical results for a sharp cutoff flow [12], except for the location of the line $1/\eta = 0$ which is nonuniversal.

Finally, we note that the qualitative picture detailed above persists in dimensions higher than 4. Therefore quantum gravity in higher dimensions may well be formulated as a fundamental theory. This consideration is
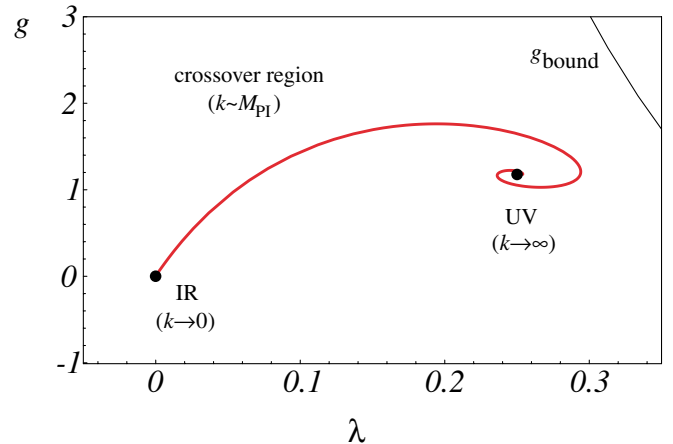

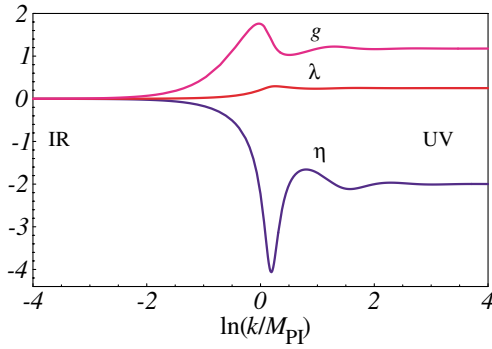FIG. 2 (color online). The separatrix in four dimensions.

FIG. 3 (color online). Running of couplings along the separatrix.

also of interest for recent phenomenological scenarios based on gravity in extra dimensions. In higher dimensions, higher dimensional operators beyond the truncation (2) are likely to be more relevant than in lower ones. Consequently, the projection of the full flow onto the Einstein-Hilbert truncation (2) and the respective domain of validity are more sensitive to the cutoff. A first analysis for specific cutoffs in $d > 4$ has revealed that the fixed point exists up to some finite dimension, where λ_* reached the boundary of the domain of validity [12]. No definite conclusion could be drawn for larger dimensions. Based on an optimized flow, the main new result here is that a nontrivial ultraviolet fixed point exists within the domain of the validity of (4) for arbitrary dimension. Furthermore, the fixed point is smoothly connected to its $4d$ counterpart and shows only a weak dependence on the gauge fixing parameter. In the limit of arbitrarily large dimensions, this leads to $\lambda_* \rightarrow 1/2$ and $g_* \rightarrow c_d/(2d^2)$ in the explicit solution (9). Similar results are obtained for an arbitrary gauge fixing parameter. Note that λ_* approaches its boundary value very slowly, increasing from $1/4$ to 0.4 for d ranging from 4 to 40. Using (9) and the definition for g_{bound}, we derive $g_{\text{bound}}(\lambda_*)/g_* = (2d)/(d - 2)$ at the fixed point. Hence, g_* stays clear by at least a factor of 2 from the boundary where $1/\eta$ vanishes, and the ultraviolet fixed point resides within the domain of validity of (4) even in higher dimensions. This stability of the fixed point also strengthens the result in lower dimensions, including $d = 4$.

In summary, we have found analytic results for the flow and a nontrivial ultraviolet attractive fixed point of quantum gravity. Maximal reliability of the present truncation is guaranteed by the underlying optimization. The fixed point is remarkably stable with only a mild dependence on the gauge fixing parameter. Furthermore, it extends to dimensions higher than $d = 4$, a region which previously has not been accessible. The qualitative structure of the phase diagram in the Einstein-Hilbert truncation is equally robust. In four dimensions the results match

with earlier numerical findings based on different cutoffs. Hence it is likely that the ultraviolet fixed point exists in the full theory. We expect that the analytical form of the flow, crucial for the present analysis, is equally useful in extended truncations. If the above picture persists in these cases, gravity is nonperturbatively renormalizable in the sense of Weinberg's asymptotic safety scenario.

———

*Email address: Daniel.Litim@cern.ch

[1] S. Weinberg, in *General Relativity: An Einstein Centenary Survey*, edited by S.W. Hawking and W. Israel (Cambridge University Press, Cambridge, UK, 1979), p. 790.

[2] G. Parisi, Nucl. Phys. **B100**, 368 (1975); **B254**, 58 (1985); K. Gawedzki and A. Kupiainen, Nucl. Phys. **B262**, 33 (1985); Phys. Rev. Lett. **54**, 2191 (1985); **55**, 363 (1985); B. Rosenstein, B. Warr, and S. H. Park, Phys. Rep. **205**, 59 (1991); C. de Calan, P. A. Faria da Veiga, J. Magnen, and R. Seneor, Phys. Rev. Lett. **66**, 3233 (1991).

[3] J. Berges, N. Tetradis, and C. Wetterich, Phys. Rep. **363**, 223 (2002); J. Polonyi, Centr. Eur. Sci. J. Phys. **1**, 1 (2002); D. F. Litim and J. M. Pawlowski, hep-th/9901063.

[4] D. F. Litim, Phys. Lett. B **486**, 92 (2000); Phys. Rev. D **64**, 105007 (2001); Int. J. Mod. Phys. A **16**, 2081 (2001).

[5] D. F. Litim, Nucl. Phys. **B631**, 128 (2002); J. High Energy Phys. 11 (2001) 059; Acta Phys. Slov. **52**, 635 (2002).

[6] M. Reuter, Phys. Rev. D **57**, 971 (1998).

[7] O. Lauscher and M. Reuter, Phys. Rev. D **65**, 025013 (2002).

[8] O. Lauscher and M. Reuter, Classical Quantum Gravity **19**, 483 (2002); Phys. Rev. D **66**, 025026 (2002).

[9] M. Reuter and C. Wetterich, Phys. Rev. D **56**, 7893 (1997); F. Freire, D. F. Litim, and J. M. Pawlowski, Phys. Lett. B **495**, 256 (2000).

[10] D. F. Litim and J. M. Pawlowski, Phys. Lett. B **435**, 181 (1998); J. High Energy Phys. 09 (2002) 049; Phys. Lett. B **546**, 279 (2002); J. M. Pawlowski, Acta Phys. Slov. **52**, 475 (2002); Int. J. Mod. Phys. A **16**, 2105 (2001); H. Gies, Phys. Rev. D **66**, 025006 (2002); **68**, 085015 (2003); V. Branchina, K. A. Meissner, and G. Veneziano, Phys. Lett. B **574**, 319 (2003).

[11] W. Souma, Prog. Theor. Phys. **102**, 181 (1999).

[12] M. Reuter and F. Saueressig, Phys. Rev. D **65**, 065016 (2002).

[13] R. Percacci and D. Perini, Phys. Rev. D **67**, 081503 (2003); **68**, 044018 (2003).

[14] P. Forgacs and M. Niedermaier, hep-th/0207028; M. Niedermaier, J. High Energy Phys. 12 (2002) 066; Nucl. Phys. **B673**, 131 (2003).

[15] H. W. Hamber, Phys. Rev. D **61**, 124008 (2000); **45**, 507 (1992).

[16] A. Bonanno and M. Reuter, Phys. Rev. D **62**, 043008 (2000); **65**, 043508 (2002); Phys. Lett. B **527**, 9 (2002); E. Bentivegna, A. Bonanno, and M. Reuter, J. Cosmol. Astropart. Phys. **01**, 001 (2004).