Microscopic Origin of the Next-Generation Fractional Quantum Hall Effect

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Most of the fractions observed to date belong to the sequences $\nu = n/(2pn \pm 1)$ and $\nu = 1 - n/(2pn \pm 1)$, n and p integers, understood as the familiar *integral* quantum Hall effect of composite fermions. These sequences fail to accommodate, however, many fractions such as $\nu = 4/11$ and $5/13$, discovered recently in ultrahigh mobility samples at very low temperatures. We show that these "next generation" fractional quantum Hall states are accurately described as the *fractional* quantum Hall effect of composite fermions.

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Electrons confined in two dimensions, when subjected to a strong magnetic field, form a quantum fluid that exhibits the remarkable phenomena of integral and fractional quantum Hall effects [1,2], namely, quantized Hall resistance plateaus at $R_H = h/\nu e^2$ with integral and fractional values of ν. The integral quantum Hall effect (IQHE) is explained as a property of uncorrelated electrons, resulting from a quantization of the kinetic energy of electrons into Landau levels in the presence of a magnetic field. The fractional quantum Hall effect (FQHE), on the other hand, is a manifestation of a strongly correlated quantum fluid. At very strong magnetic fields, electrons fall into the lowest Landau level (LL) and the physics is entirely governed by the repulsive Coulomb interaction. Many essential properties of this quantum fluid can be explained by postulating that electrons in the lowest LL minimize their interaction energy by capturing an even number ($2p$) of quantized vortices each to turn into composite fermions (CF's) [3], which experience an effective magnetic field and form their own Landau-like levels, termed "CF-quasi-Landau levels." The number of occupied CF-quasi-Landau levels, ν^*, is related to the filling factor of the lowest electron LL, ν, according to the formula $\nu = \nu^*/(2p\nu^* \pm 1)$. In particular, the IQHE of composite fermions ($\nu^* = n$) provides an explanation for the FQHE of electrons at $R_H = h/\nu e^2$, with $\nu = n/(2pn \pm 1)$.

The observation of fractions such as $4/11$ and $5/13$ [4–7] points to new physics beyond the integral quantum Hall effect of composite fermions. It has been appreciated that the residual interaction between composite fermions can, in principle, cause such fractions [3,8], in the same way as the interaction between electrons produces the FQHE. For example, consider composite fermions carrying two vortices ($p = 1$). If they were completely noninteracting, only $\nu = n/(2n \pm 1)$ would be obtained. However, there *is* a weak residual interaction between composite fermions. If it happens to be of a form that produces a *fractional* QHE of composite fermions at

$$\nu^* = 1 + \bar{\nu} = 1 + \frac{m}{2m \pm 1} \tag{1}$$

(m = integer), then that would result in new *electron* fractions in the range $2/5 > \nu > 1/3$, given by

$$\nu = \frac{3m \pm 1}{8m \pm 3}. \tag{2}$$

These include the newly observed fractions [9]. (Of course, many more fractions can be constructed in this manner [8].)

This qualitative picture is intuitively appealing and indicates that the next generation FQHE is possible at least for some model interaction between composite fermions. However, to confirm this scenario, it is important to carry out quantitative tests to determine if the FQHE of composite fermions will occur for the *actual* residual interaction between composite fermions, a remnant from the Coulomb interaction between electrons. A FQHE at ν requires that the state here be incompressible, that is, have a uniform ground state with a gap to excitations. One can ascertain incompressibility from either exact numerical diagonalization on small systems, or "CF diagonalization" [10] (outlined below) for larger systems. Extensive studies of $\nu = 4/11$ as a function of the number of electrons, N, have found that the state is incompressible for $N = 12$ and 20 but compressible for $N = 8$, 16, and 24 [10–14]. While the message was mixed, it was on the whole interpreted to mean that the results do not support, in the thermodynamic limit, a fully spin-polarized FQHE at $\nu = 4/11$ [10]. That conclusion, however, is incompatible with experiments [4,5], which show a clear evidence for a fully polarized FQHE at $\nu = 4/11$. We explain below the origin of the intriguing behavior for finite systems, why it does *not* rule out incompressibility in the thermodynamic limit (contrary to our previous assertion), and then go on to write explicit wave functions to confirm, quantitatively, that the new fractions indeed are well described as the FQHE of composite fermions.

The calculations below consider N electrons on the surface of a sphere, moving under the influence of a radial magnetic field B produced by a Dirac monopole of strength Q at the center, which produces a net magnetic flux of $2Q$, in units of $\phi_0 = hc/e$, through the surface.

($2Q$ is an integer according to Dirac's quantization condition.) CF diagonalization refers to determining the low-energy spectrum by numerically diagonalizing the Hamiltonian in the correlated CF basis:

$$\{\mathcal{P}_{\rm LLL}\Phi_1^{2p}\Phi_{Q^*}^{\alpha}\}. \tag{3}$$

Here $\{\Phi_{Q^*}^{\alpha}\}$ is an orthogonal basis of N-electron states at $Q^* = Q - p(N-1)$. We are interested in the filling factor range $2/5 > \nu > 1/3$, for which the CF filling at ν^* lies between one and two (with $p = 1$). We include all electron basis states at Q^* which have the lowest LL completely occupied and the second partially occupied. Φ_1 is the wave function for a fully occupied Landau level, and $\mathcal{P}_{\rm LLL}$ denotes projection into the lowest Landau level (LLL). The basis states in Eq. (3) are in general not linearly independent. We extract an orthogonal basis following the Gram-Schmidt procedure and then diagonalize the Coulomb Hamiltonian to find eigenstates and eigenenergies. The Hamiltonian matrix elements are evaluated by the Metropolis Monte Carlo method. (All energies are quoted in units of $e^2/\epsilon l_0$, where ϵ is the dielectric constant of the background semiconductor, and $l_0 \equiv \sqrt{\hbar c/eB}$ is the magnetic length.) The statistical uncertainty is determined by performing many (~ 10) Monte Carlo runs, with 0.8–1.0×10^6 iterations in each run. The basis states are, by construction, in the lowest Landau level, so our results provide strict variational bounds on the ground state energy in the limit $B \to \infty$. The eigenstates have definite orbital angular momentum, L, with $L = 0$ for uniform ground states. The ground state obtained by CF diagonalization will be denoted Ψ_ν^0. Details of lowest LL projection and diagonalization can be found in the literature [10,15]. The state at filling factor ν of Eq. (2) is obtained at flux values given by [10]

$$2Q = \frac{8m \pm 3}{3m \pm 1}N - \frac{12m \pm (m^2+3)}{3m \pm 1}, \tag{4}$$

which ensures $\lim_{N\to\infty} N/2Q = \nu = (3m \pm 1)/(8m \pm 1)$. It has been shown in the past that the CF diagonalization method produces essentially the same results as exact diagonalization (see, for example, Ref. [16]).

We begin by pointing out the flaw in the reasoning of Ref. [10] that led to the conclusion that the fully spin-polarized state at $\nu = 4/11$, etc. is compressible in the thermodynamic limit. It was implicitly assumed in Ref. [10] that if a state is incompressible in the thermodynamic limit, then all its finite size realizations must also be incompressible. This criterion was used because there was no known exception to it for FQHE in the lowest Landau level, at fractions of the form $\nu = n/(2pn \pm 1)$. However, the criterion is not universally valid, and FQHE states in *higher* LL's provide an explicit counterexample. Consider the electron state at $\nu^* = 1 + \bar{\nu}$, with $\bar{N}$ particles forming a state with filling factor $\bar{\nu}$ in the second LL. Given that the filled lowest LL is inert, one might expect that the state in the second LL at $\bar{\nu}$ is quite similar to the corresponding state at filling factor $\bar{\nu}$ in the lowest LL, but, in reality, there are striking differences between the two [17,18], for reasons not fully understood at present. Consider the example of $\bar{\nu} = 1/3$. For the 1/3 state in the lowest LL, the system is incompressible for all N, whereas for the 1/3 state in the second LL, the ground state is compressible ($L \neq 0$) for $\bar{N} = 3$ and 5, and almost compressible for $\bar{N} = 7$. (See Ref. [17] and Table I. The gap for $\bar{N} = 7$ is a factor of 37 smaller than the gap at $\bar{N} = 6$.) Furthermore, the ground state wave functions at 1/3 in the lowest and second LL's are rather different; the largest overlap between them is obtained for seven particles, which is only ~ 0.6 [17]. Because of such strong fluctuations as a function of N it was initially thought [18] that exact diagonalization studies *rule out* FQHE at $\bar{\nu} = 1/3$ in the second LL. Study of bigger systems revived the possibility of incompressibility in the thermodynamic limit [17], and FQHE at 1/3 in the second LL has been observed experimentally [19], albeit with a small gap of ~ 100 mK.

Could something similar be happening at the newly observed fractions? That would be quite natural from the CF perspective, which relates the new fractions (e.g., $\nu = 4/11$) to the FQHE of composite fermions in *higher* CF-quasi-LL's (e.g., $\nu^* = 4/3$). We now proceed to investigate the issue quantitatively. To begin with, Table I gives the gaps at several values of ν given by Eq. (2), obtained by CF diagonalization. (No gap is given when the ground state is not uniform.) The behavior is remarkably analogous to that at $\nu^* = 1 + \bar{\nu}$ [Eq. (1)]. For example, including the electrons in the lowest LL, the state at $\nu^* = 4/3$ is compressible for $N = 8$ and 16 particles ($\bar{N} = 3$ and 5) and almost compressible for $N = 24$ ($\bar{N} = 7$); these match the particle numbers for which $\nu = 4/11$ has been found to be compressible. The states at 5/13 and 7/19 are similar to those at 5/3 and 7/5. The analogy between ν and ν^* strongly suggests that, in spite of finite

TABLE I. The gaps at ν and ν^*, determined from CF and exact diagonalization, respectively, at several filling factors. Only the gaps of incompressible states are shown. N is the total number of particles and $\bar{N}$ is the number of particles in the second LL for the state at ν^*. The gaps are quoted in units of $e^2/\epsilon l_0$. The statistical uncertainty from Monte Carlo is shown in parentheses.

ν	ν^*	$2Q$	N	$\bar{N}$	Gap (ν)	Gap (ν^*)
		18	8	3	$\cdots$	$\cdots$
$\frac{4}{11}$	$\frac{4}{3}$	29	12	4	0.010(1)	0.035
		40	16	5	$\cdots$	$\cdots$
		51	20	6	0.006(2)	0.024
		62	24	7	$\cdots$	0.000 64
$\frac{5}{13}$	$\frac{5}{3}$	33	14	6	0.003(1)	0.035
		46	19	8	$\cdots$	$\cdots$
		20	9	4	$\cdots$	$\cdots$
$\frac{7}{19}$	$\frac{7}{5}$	39	16	6	0.006(2)	0.016
		58	23	8	$\cdots$	0.0018

size fluctuations, the state at ν is incompressible in the thermodynamic limit. It would be desirable to study systems at $\nu = (3m \pm 1)/(8m \pm 3)$ larger than those reported here and in Ref. [10], but that is not possible with the present day computers.

We now concentrate on those particle numbers for which the states are incompressible, which we believe contain the physics of incompressibility in the thermodynamic limit. A secure understanding of the origin of a FQHE state rests on identifying an accurate wave function that reveals its microscopic physics. Wave functions for the new FQHE states can be constructed based on the above physical picture following the standard procedure, which allows for a microscopic test of the scenario. For the ground state at $\nu = (3m \pm 1)/(8m \pm 3)$ the trial wave function is given by

$$\Psi_\nu^{\rm tr} = \mathcal{P}_{\rm LLL}\Phi_1^2\Phi_{\nu^*}, \tag{5}$$

where Φ_{ν^*} is the $L = 0$ Coulomb ground state at $\nu^* = 1 + m/(2m \pm 1)$, obtained by exact diagonalization. Because multiplication by Φ_1^2 attaches two vortices to each electron to convert it into a composite fermion, the wave function $\Psi_\nu^{\rm tr}$ is interpreted as the FQHE of composite fermions at $\nu^* = 1 + m/(2m \pm 1)$. (Although amenable to an intuitive interpretation through composite fermions, the actual wave function is extremely intricate.) As another reference point, we also present results for the trial wave function

$$\Psi_\nu^{\prime\rm tr} = \mathcal{P}_{\rm LLL}\Phi_1^2\Phi'_{\nu^*}, \tag{6}$$

where Φ'_{ν^*} is obtained by placing in the second LL the Coulomb ground state at $\bar{\nu} = m/(2m \pm 1)$ of the *lowest* Landau level. $\Psi_\nu^{\rm tr}$ is derived from the $m/(2m \pm 1)$ state in the second LL, whereas $\Psi_\nu^{\prime\rm tr}$ is analogous to the $m/(2m \pm 1)$ state in the lowest LL.

Because Ψ_ν^0 are very accurate [16], the overlaps and energies given in Table II establish that $\Psi_\nu^{\rm tr}$ are also very accurate. (The overlap of 0.86 is significantly large for a system with $N = 20$ particles.) A direct comparison with *exact* results is possible for the 12 particle state at $\nu = 4/11$. For this system, the energies from the CF theory, $E^0 = -0.441\,05(9)$ and $E^{\rm tr} = -0.440\,88(4)$, deviate from the exact energy, $-0.441\,214$ [14] by 0.04% and 0.08%. The level of agreement is highly significant for a system with 12 particles and similar to that for the accepted trial wave functions for the ordinary FQHE at $\nu = n/(2pn \pm 1)$. It gives an unambiguous indication, at a microscopic level, of a direct connection between the physics of the FQHE at ν and ν^*.

Thus, the analogy between the FQHE in the lowest LL at ν [Eq. (2)] and the FQHE in the second LL at $\nu^* = 1 + \bar{\nu}$ [Eq. (1)] not only explains the qualitative behavior as a function of N but also produces accurate wave functions for the *incompressible* ground states at ν. These facts taken together give us confidence that the new fractions are a manifestation of the FQHE of composite fermions.

Even though the finite system incompressible states help us confirm the physics of the new fractions, an accurate determination of the excitation gaps is not possible because we do not have enough points for a reliable extrapolation to the thermodynamic limit. The gap in Table I for the largest available incompressible system at a given filling can be taken as a very crude estimate for the thermodynamic gap. The gaps for the new FQHE states are more than an order of magnitude smaller than the gaps at $\nu = 1/3$ and $2/5$ in the lowest LL, explaining why the new FQHE states are much more fragile, more readily destroyed by disorder or thermal fluctuations, than the ordinary FQHE states at $1/3$ and $2/5$, in spite of their close proximity. The smallness of the gap is illustrative of the fact that the residual interaction between composite fermions is much weaker than the interaction between electrons. (All gaps being compared are theoretical gaps, without including the effects of finite thickness or disorder.)

There has been much recent theoretical work on the new FQHE states. Pairing of composite fermions has been advanced [20] as an alternative possible mechanism for the next generation fractions, and another theoretical paper [21] has studied the FQHE of composite fermions using a Hamiltonian approach [22]. However, the quantitative accuracy of the methods used in these works has not been established at a level required for the issue of stability of the delicate new FQHE states, and, in particular, neither of these approaches constructs an explicit wave function which can be directly compared with the exact ground state wave function known for small systems. An effective field theory approach [23] is also not best suited to address the stability of a FQHE state, although it may illuminate certain properties thereof assuming it exists.

We discuss briefly certain approximations made in our work. (i) Our main assumption is the neglect of mixing between the CF-quasi-LL's. Our preliminary studies, which relax the assumption by enlarging the basis (by allowing for LL mixing at Q^*), find that the corrections are very small and do not change the qualitative results. Indeed, the fact that Ψ_ν^0 is very close to the exact ground state is indicative of the insignificance of CF-quasi-LL mixing. (ii) We are also neglecting, throughout, a mixing between *electronic* Landau levels at Q, but that is presumably negligible at the highest magnetic fields where the next generation FQHE has been observed [4,5]. (iii) We have also studied the correction due to a finite transverse thickness of the electron system, which modifies the form of the effective two-dimensional interaction between electrons; it lowers all energies but does not alter significantly either the qualitative nature of the ground state or the form of the ground state wave function. (iv) The Zeeman energy has been assumed to be frozen. Spin related physics can also generate new fractions, associated with partially spin-polarized states, which would be observable at relatively low magnetic fields [24]. The

TABLE II. Comparison of trial wave functions $\Psi_\nu^{\rm tr}$ and $\Psi_\nu^{\prime\rm tr}$ with Ψ_ν^0 for several incompressible states at three filling factors. (See the text for definitions.) The overlaps are defined as $\mathcal{O} = \langle\Psi_\nu^0|\Psi_\nu^{\rm tr}\rangle/\sqrt{\langle\Psi_\nu^0|\Psi_\nu^0\rangle\langle\Psi_\nu^{\rm tr}|\Psi_\nu^{\rm tr}\rangle}$ and $\mathcal{O}' = \langle\Psi_\nu^0|\Psi_\nu^{\prime\rm tr}\rangle/\sqrt{\langle\Psi_\nu^0|\Psi_\nu^0\rangle\langle\Psi_\nu^{\prime\rm tr}|\Psi_\nu^{\prime\rm tr}\rangle}$. E^0, $E^{\rm tr}$, and $E^{\prime\rm tr}$ are the Coulomb energies per particle for Ψ_ν^0, $\Psi_\nu^{\rm tr}$, and $\Psi_\nu^{\prime\rm tr}$, respectively. E^0 were reported in Ref. [10].

ν	N	$\mathcal{O}$	$\mathcal{O}'$	E^0	$E^{\rm tr}$	$E^{\prime\rm tr}$
$\frac{4}{11}$	12	0.993(2)	0.51(1)	−0.441 05(9)	−0.440 88(4)	−0.436 70(4)
	20	0.86(1)	0.278(8)	−0.430 27(5)	−0.429 75(5)	−0.427 05(6)
$\frac{5}{13}$	14	0.973(1)	0.365(3)	−0.444 00(9)	−0.443 74(9)	−0.439 51(3)
$\frac{7}{19}$	16	0.990(4)	0.009(2)	−0.438 08(4)	−0.438 06(4)	−0.432 83(5)

states observed in Refs. [4,5] are insensitive to changes in the Zeeman energy and survive to very high magnetic fields, indicating that they are fully spin polarized. An early hint of fractions outside the sequences $n/(2pn \pm 1)$ in very low density samples [7] may involve partial spin reversal.

An extension of the above analogy between FQHE at ν and ν^* has implications for future fractions. There is good evidence [25] that the Coulomb interaction does not stabilize FQHE of electrons at $\nu = \bar{n} + m/(2m \pm 1)$ for $\bar{n} > 1$, but fractions like $\bar{\nu} = 1/5$ and $\bar{\nu} = 4/5$ may occur in the third LL. Assuming similar behavior for composite fermions, this would rule out fully spin-polarized FQHE at electron fractions of the form $[(2\bar{n}+1)m \pm \bar{n}]/[4(\bar{n}+1)m \pm (2\bar{n}+1)]$ with $\bar{n} > 1$, but leave open the possibility of FQHE at apparently more complicated fractions like $\nu = 11/27$ and $14/33$ in the filling factor range $2/5 < \nu < 3/7$. Charge density waves of various types are also predicted to occur for certain filling factors in higher quasi-LL's of composite fermions [26].

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[1] K. von Klitzing, G. Dorda, and M. Pepper, Phys. Rev. Lett. **45**, 494 (1980).
[2] D. C. Tsui, H. L. Stormer, and A. C. Gossard, Phys. Rev. Lett. **48**, 1559 (1982).
[3] J. K. Jain, Phys. Rev. Lett. **63**, 199 (1989).
[4] W. Pan, H. L. Stormer, D. C. Tsui, L. N. Pfeiffer, K.W. Baldwin, and K.W. West, Phys. Rev. Lett. **90**, 016801 (2003).
[5] W. Pan, H. L. Stormer, D. C. Tsui, L. N. Pfeiffer, K.W. Baldwin, and K.W. West, Int. J. Mod. Phys. B **16**, 2940 (2002).
[6] J. H. Smet, Nature (London) **422**, 391 (2003).
[7] Also see V. J. Goldman and M. Shayegan, Surf. Sci. **229**, 10 (1990).
[8] J. K. Jain, Phys. Rev. B **41**, 7653 (1990).
[9] We consider here only fully spin-polarized states, i.e., effectively "spinless" fermions, so $\nu^* = 1 + \bar{\nu}$ has lowest LL completely full and a filling $\bar{\nu}$ in the *second* LL. In actual experiments, the filling factor $\bar{\nu}$ in the second second LL corresponds to the total filling factor $\nu = 2 + \bar{\nu}$.
[10] S. S. Mandal and J. K. Jain, Phys. Rev. B **66**, 155302 (2002).
[11] Exact spectra for eight particles in the filling factor range between $1/3$ and $2/5$ were first presented in Ref. [12]. Reference [13] noted that the eight particle state at $\nu = 4/11$ was not incompressible. Recently, Rezayi [14] has obtained exact results for 12 particles at 4/11 (which is incompressible). Systems with up to 24 particles were studied in Ref. [10] by CF diagonalization.
[12] S. He, X. C. Xie, and F. C. Zhang, Phys. Rev. Lett. **68**, 3460 (1992).
[13] A. Wójs and J. J. Quinn, Phys. Rev. B **61**, 2846 (2000).
[14] E. H. Rezayi (unpublished).
[15] J. K. Jain and R. K. Kamilla, Int. J. Mod. Phys. B **11**, 2621 (1997).
[16] J. K. Jain, C.-C. Chang, G. S. Jeon, and M. R. Peterson, Solid State Commun. **127**, 805 (2003)
[17] N. d'Ambrumenil and A. M. Reynolds, J. Phys. C **21**, 119 (1988).
[18] F. D. M. Haldane, in *The Quantum Hall Effect*, edited by R. E. Prange and S. M. Girvin (Springer-Verlag, New York, 1987).
[19] W. Pan, J.-S. Xia, V. Shvarts, D. E. Adams, H. L. Stormer, D. C. Tsui, L. N. Pfeiffer, K.W. Baldwin, and K.W. West, Phys. Rev. Lett. **83**, 3530 (1999).
[20] A. Wójs, K.-S. Yi, and J. J. Quinn, cond-mat/0312290; J. J. Quinn, A. Wójs, and K.-S. Yi, Phys. Lett. A **318**, 152 (2003).
[21] M. O. Goerbig, P. Lederer, and C. Morais Smith, Phys. Rev. B **69**, 155324 (2004).
[22] G. Murthy and R. Shankar, Rev. Mod. Phys. **75**, 1101 (2003).
[23] A. Lopez and E. Fradkin, Phys. Rev. B **69**, 155322 (2004); M. Flohr and K. Osterloh, Phys. Rev. B **67**, 235316 (2003).
[24] C.-C. Chang, S. S. Mandal, and J. K. Jain, Phys. Rev. B **67**, 121305(R) (2003).
[25] G. Gervais *et al.*, cond-mat/0402169, and references therein.
[26] S.-Y. Lee, V.W. Scarola, and J. K. Jain, Phys. Rev. Lett. **87**, 256803 (2001); Phys. Rev. B **66**, 085336 (2002); M. O. Goerbig, P. Lederer, and C. Morais Smith, cond-mat/0403016.