1 + 1 Dimensional Compactifications of String Theory

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We argue that stable, maximally symmetric compactifications of string theory to 1 + 1 dimensions are in conflict with holography. In particular, the finite horizon entropies of the Rindler wedge in 1 + 1 dimensional Minkowski and anti–de Sitter space, and of the de Sitter horizon in any dimension, are inconsistent with the symmetries of these spaces. The argument parallels one made recently by the same authors, in which we demonstrated the incompatibility of the finiteness of the entropy and the symmetries of de Sitter space in any dimension. If the horizon entropy is either infinite or zero, the conflict is resolved.

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Introduction.—The holographic principle [1,2] has become one of the most important ideas in quantum gravity. One may wonder whether it constrains the possible backgrounds in string theory, or any consistent theory that includes gravity. Indeed, in a recent paper [3] we have argued that the symmetries of de Sitter space are incompatible with the holographic requirement of finite entropy of a causal patch. We further argued that the incompatibility shows up in the very long time behavior of inflating spaces, perhaps as an inevitable instability.

In this Letter, we will use the same argument to constrain the set of possible maximally symmetric compactifications of string theory to two dimensions, at least in cases where they admit a geometric description (we thank Juan Maldacena for suggesting the application of the argument to AdS_2). We will show that consistency with holography requires at least three space-time dimensions.

In [3], the authors noted the fact that, in all representations of the de Sitter group, the boost generator (which generates time translations for the static patch) has a continuous spectrum. However, the area of the de Sitter horizon is finite, which implies a finite entropy for the quantum de Sitter Hamiltonian, and finite entropy is inconsistent with a continuous spectrum. Therefore, pure, eternal de Sitter space is inconsistent with holography. The simplest explanation is that all de Sitter solutions in string theory are at best metastable, with a finite lifetime followed by decay to a supersymmetric vacuum.

We can apply the same logic to 1 + 1 dimensional Lorentz and anti–de Sitter (AdS) invariant compactifications of string theory. The area of the horizon of the Rindler wedge of 1 + 1 dimensional Minkowski space is simply the volume of the compact internal space and, hence, is finite. However, the 1 + 1 dimensional Poincaré algebra has the same commutation relation used in [3] to prove continuity of the spectrum. Therefore, by the same argument, the Rindler Hamiltonian is continuous, which is inconsistent with the finite area of the horizon. An identical argument can be made for 1 + 1 dimensional anti–de Sitter space. Our conclusion is that maximally symmetric compactifications of string theory to 1 + 1 dimensions in which the horizon area is macroscopic are never stable. One possible resolution is that the volume of the compact space is infinite; another is that it is very small and the entropy is zero. In the latter case, there is only order one state in the Hilbert space, and the symmetry argument does not apply.

It is also the case that supersymmetric compactifications to 1 + 1 dimensions will typically contain massless moduli, which in two dimensions have an IR quantum instability. The expectation values of such moduli will quickly spread to infinity (e.g., if an IR cutoff is suddenly removed), which may account for the instability in the size of the compact manifold. Similar arguments were made in [4].

The plan of the Letter is as follows: First, we review the formalism of thermofield dynamics and its connection to the Rindler wedge of Minkowski space. In the next section, we prove (using the technique of [3]) that the Rindler Hamiltonian is continuous. The same proof applies to the Hamiltonian of the "Rindler" wedge of AdS. Finally, we discuss the field theoretic instability, non-Lorentz invariant compactifications to 1 + 1, and conclude.

Thermofield dynamics.—Thermofield theory was invented [5] in the context of the many body theory for the purpose of simplifying the calculation of real time correlation functions in finite temperature systems. Its connection with black holes was realized by Israel [6], and elaborated in the holographic context by Maldacena [7]. In the thermofield formalism, one takes the tensor product of two copies of the original field theory, labeled by 1,2. The two copies are decoupled, and the total Hamiltonian is

$$H_{tf} \equiv H \otimes I - I \otimes H, \tag{1}$$

where H is the Hamiltonian for the original theory and I is the identity operator. We then construct the entangled state

$$|\psi\rangle = \frac{1}{\sqrt{Z}} \sum_i e^{-(1/2)} \beta E_i |E_i, E_i\rangle, \tag{2}$$

where $|E_i, E_j\rangle = |E_i\rangle \otimes |E_j\rangle$, and $|E_i\rangle$ are energy eigenstates. The state $|\psi\rangle$ is a particular eigenvector of H_{tf} with eigenvalue zero. Correlations between subsystems 1 and 2 are due to the entanglement in $|\psi\rangle$.

Operators which belong to subsystem 1 have the form $A \otimes I$, and will be denoted A_1. Operators associated with subsystem 2 are defined in a similar manner, except with an additional rule of Hermitian conjugation:

$$A_2 \equiv I \otimes A^{\dagger}. \tag{3}$$

Standard thermal correlation functions may be written as expectation values:

$$\langle\psi|A_1(0)B_1(t)|\psi\rangle. \tag{4}$$

As can be easily seen from the form of $|\psi\rangle$, (4) is simply the thermal expectation value of $A(0)B(t)$, evaluated in a thermal density matrix at inverse temperature β. The state counting entropy observed in subsystem 1 is the entropy of entanglement of the state $|\psi\rangle$. We can define correlators involving both subsystems; for example,

$$\langle\psi|A_1(0)B_2(t)|\psi\rangle. \tag{5}$$

In the finite temperature AdS/conformal-field-theory correspondence, (5) has a simple interpretation [7]: It corresponds to a correlator between operators on the two disconnected boundaries of the space-time. It is not hard to see that one can compute (5) by analytically continuing (4):

$$\langle\psi|A_1(0)B_2(t)|\psi\rangle = \langle\psi|A_1(0)B_1(-t - i\beta/2)|\psi\rangle. \tag{6}$$

Rindler space.—We will now consider the relationship between thermofield dynamics and quantum field theory in spaces with horizons. The simplest example is Rindler space. One plus one dimensional Minkowski space can be divided into four quadrants (see Fig. 1). Quadrant I is Rindler space, and can be described by the metric

$$ds^2 = r^2 dt^2 - dr^2, \tag{7}$$

where r is proper distance from the origin, and t is the dimensionless Rindler time. The Rindler quadrant may be described by the Unruh thermal state with temperature

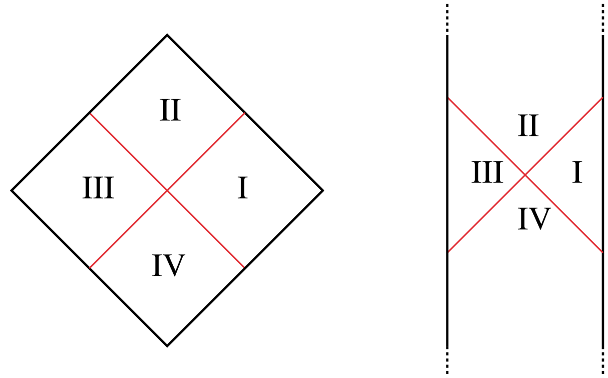

FIG. 1 (color online). On the left, the conformal diagram for Minkowski space, showing the Rindler wedges I and III. In Rindler coordinates, continuing from region I to region III involves shifting time by $i\beta/2$, in accord with (6). On the right, the "Rindler" wedge of AdS_2.

$$T_{\rm rind} = \frac{1}{2\pi} = \frac{1}{\beta_{\rm rind}}. \tag{8}$$

Quadrant III is a copy of quadrant I, and can be precisely identified with the other half of the thermofield double. To see this, we first of all note that the Rindler Hamiltonian is the boost generator. Since the Minkowski vacuum is boost invariant, it is an eigenvector of the boost generator with vanishing eigenvalue. Furthermore, the Minkowski vacuum is an entangled state of the degrees of freedom in the two quadrants I and III. Finally, it is well known that when the density matrix for quadrant I is obtained by tracing over III the result is a thermal state at the Rindler temperature.

Recall that the usual Minkowski variables X^0, X^1 are related to the Rindler coordinates by

$$X^0 = r \sinh t, \qquad X^1 = r \cosh t. \tag{9}$$

Since the inverse Rindler temperature (8) is 2π, the continuation in Eq. (6) is

$$t \to t - i\pi, \tag{10}$$

or, from (9), $X^\mu \to -X^\mu$. As expected, the thermofield continuation takes quadrant I to quadrant III.

In any dimension other than $1 + 1$, the area of the horizon—and therefore the entropy of Rindler space—is infinite. However, if we consider the space $\mathcal{R}_{1,1} \times \mathcal{M}_8$, where $\mathcal{M}_8$ is any eight-dimensional compact manifold, the area of the Rindler horizon is nothing but vol($\mathcal{M}_8$), and therefore the entropy of the Rindler wedge must be finite.

A very similar construction produces the Rindler wedge of anti–de Sitter. Start with the coordinates in which Euclidean AdS is a ball: $ds^2 = d\rho^2 + \sinh^2 \rho d\Omega^2$. Continuing the azimuthal angle of the sphere $\phi \to it$ gives a metric covering one-quarter of the AdS hyperboloid (see Fig. 1). As in flat space, the horizon area is finite only for the case of AdS_2.

Continuity of the spectrum.—It is a fact that any system with finite entropy must have a discrete spectrum; actually, the statement is stronger: There must be a finite number of states below any given energy E, so the spectrum cannot have discrete accumulation points (note that this does *not* imply that the Hilbert space is finite — in general, the level spacing $\delta E \sim Te^{-S}$, at least for $T \gg \delta E$). In the interest of brevity, we will not give a proof here (one can be found in the arxiv version of this Letter).

Therefore, if the entropy of the Rindler wedge is finite, then the Rindler Hamiltonian H_R must have a discrete spectrum. The implications for the thermofield Hamiltonian H_{tf} are weaker. The spectrum of this operator consists of the differences of eigenvalues of H_R, namely $E_i - E_j$. If the spectrum of H_R is discrete, this set of numbers need not be, but is certainly countable. In the next section, we will prove that the spectrum of the full thermofield Hamiltonian H_{tf} is not countable, and, hence, that the entropy of the thermal ensemble defined using H_R is infinite.

Representations of the Poincaré algebra.—The Poincaré group in 1 + 1 dimensions has three generators, which satisfy the following algebra:

$$[P_0, K] = iP_1, \qquad [P_1, K] = iP_0, \qquad [P_0, P_1] = 0. \tag{11}$$

The P_i generate translations in space and time, and K is the boost generator. The choice of a Rindler wedge preserves only the generator K, which acts by generating translations in Rindler time t; in other words, the Rindler Hamiltonian is K. The generators P_i do not act on the Hilbert space of one wedge; rather, they act on the full thermofield double space by mixing degrees of freedom from the two halves. In the case of a free field theory, this means that the modes of one Rindler wedge become a linear combination of modes of both wedges under the action of the P_i; see the appendix of [3] for more details. The crucial point is that the states of the thermofield double space, namely Minkowski space, form a representation of the full group.

If we define $P_\pm = P_0 \pm P_1$, and identify the boost K with the thermofield Hamiltonian H, the algebra becomes (for the rest of the section denote the full thermofield Hamiltonian H_{tf} as H)

$$[H, P_+] = -iP_+, \qquad [H, P_-] = iP_-, \qquad [P_+, P_-] = 0. \tag{12}$$

Using (12), we wish to prove that the spectrum of H is continuous. Following [3], consider the operator $e^{iP_-}(t)$

$$e^{iP_-}(t) \equiv e^{iHt}e^{iP_-}e^{-iHt} = e^{e^{iHt}iP_-e^{-iHt}} = e^{iP_-e^{-t}}. \tag{13}$$

We will now assume that the spectrum of H is countable, and use the assumption to derive a contradiction. We have

$$|\langle\alpha|e^{iP_-}|\alpha\rangle| = 1 - \delta. \tag{14}$$

Here $|\alpha\rangle$ is some state in the Hilbert space, and $\delta > 0$ because e^{iP_-} is unitary and P_- is nonzero. Define

$$F(t) \equiv \langle\alpha|e^{iP_-(t)}|\alpha\rangle = \langle\alpha|e^{iHt}e^{iP_-}e^{-iHt}|\alpha\rangle = \langle\alpha|e^{iP_-e^{-t}}|\alpha\rangle. \tag{15}$$

From this, $F(t) \to 1$ as $t \to \infty$, and $F(0) = 1 - \delta < 1$. We will now prove that $F(t)$ is quasiperiodic (see, e.g., the appendix of [8]).

Any discrete sum of the form

$$\sum_{n=1}^{\infty} f_n e^{i\omega_n t} \tag{16}$$

is quasiperiodic if

$$\sum_{n=1}^{\infty} |f_n|^2 < \infty. \tag{17}$$

Therefore, it suffices to show that $F(t)$ can be written as a sum of this form. But, expanding the state $|\alpha\rangle$ in the energy basis (the energies ω_i are the eigenvalues of the thermofield Hamiltonian $H = H_{tf}$ and, hence, are *differences* of pairs of energies of the Rindler Hamiltonian H_R; but see the last paragraph before the section titled "Representations of the Poincaré algebra"):

$$F(t) = \sum_{n,m} f_n^* f_m \langle n|e^{iP_-}|m\rangle e^{i(\omega_n - \omega_m)t}. \tag{18}$$

Consider the sum

$$\sum_{m,n} f_n^* f_m f_m^* f_n \langle n|e^{iP_-}|m\rangle\langle m|e^{-iP_-}|n\rangle = \sum_n |f_n|^2 \sum_m |f_m|^2 \langle n|e^{iP_-}|m\rangle\langle m|e^{-iP_-}|n\rangle. \tag{19}$$

Considering the inner sum, we have (since $\sum_m \langle n|e^{iP_-}|m\rangle\langle m|e^{-iP_-}|n\rangle = 1$, and the terms are real and positive)

$$\sum_m |f_m|^2 \langle n|e^{iP_-}|m\rangle\langle m|e^{-iP_-}|n\rangle \leq 1, \tag{20}$$

and therefore

$$\sum_{m,n} f_n^* f_m f_m^* f_n \langle n|e^{iP_-}|m\rangle\langle m|e^{-iP_-}|n\rangle \leq 1. \tag{21}$$

This shows that $F(t)$ satisfies the criterion (17) and, hence, $F(t)$ is quasiperiodic. Therefore, since $F(0) < 1$, $F(t)$ cannot tend to 1 as $t \to \infty$, and we have a contradiction.

We can make almost exactly the same argument using the Rindler wedge of AdS_2. The algebra is SO(2,1) (which is identical to the dS_2 algebra) and in fact the analogy to the argument of [3] is exact. The generator of Rindler time translations is one of the boost generators of the AdS hyperboloid. Again, the argument shows the spectrum must be continuous.

This proves that H_{tf} cannot have a countable spectrum, and therefore that the E_i cannot be discrete and the entropy cannot be finite. However, the area of the Rindler horizon in $1 + 1$ dimensions is finite, and so we see there is a fundamental conflict between the holographic principle and the existence of a stable compactification of string theory to $1 + 1$ dimensional Minkowski, AdS, or dS space.

We note one possible loophole: If the representation is trivial, so that there are only vacua in the spectrum, our argument fails. In that case the generators are zero, and there is no conflict with the algebra.

String theory arguments.—Let us begin with supersymmetric compactifications of string theory. In general, there will always be massless moduli such as the overall size of the compact space. These massless degrees of freedom can be thought of as $1 + 1$ dimensional scalar fields. But it is well known that in $1 + 1$ dimensions there are logarithmic infrared divergences. For example, the two point function has a divergence of the form $\int dk/k$. While the UV divergence can of course be regulated, the IR divergence represents a true physical effect; namely, that the fields fluctuate more and more at longer and longer wavelengths. In particular, if the Hamiltonian includes a mass term $m^2\theta(-t)$, where θ is the unit step function, the expectation value $\langle\phi^2(t)\rangle$ will tend to infinity similar to t, for $t > 0$. This means that the field, once released from its confining potential, fluctuates more and more in field space. This effect indicates a decompactification of the manifold. In other words, the compactification is unstable.

This argument (which was discussed previously in [4]) is in some ways more generic than the argument presented above, because it does not rely on maximal symmetry. On the other hand, it does require that at least some of the compactification moduli are massless.

There are, of course, $1 + 1$ dimensional compactifications of string theory such as the linear dilaton vacua in which the dilaton varies linearly, either with respect to time or space. However, these obviously violate Lorentz invariance. What is more, at the weak coupling end the volume of the compact space diverges if expressed in Planck units. Other possible examples, such as pp wave in anti–de Sitter space, can be thought of as Lorentz noninvariant $1 + 1$ dimensional theories and do not contradict our conclusion.

Conclusion.—In both [3] and the present Letter, we have found that the delicate symmetries that ensure the equivalence of different observers (observer complementarity) cannot be implemented for systems of finite entropy. It seems that for these symmetries to be exact, an infinite horizon area must be available for information to spread out in. We think that this is a general rule: Exact observer complementarity is possible only if the horizon is infinite in extent.

This raises the question of finite mass black hole horizons. In this case, there is no exact symmetry between observers outside the black hole and those which fall through the horizon. Indeed, sufficiently careful measurements of tidal forces should be able to tell a freely falling observer exactly when she crosses the horizon. Only in the limit of infinite mass does the horizon become precisely undetectable.

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[1] G. 't Hooft, gr-qc/9310026.
[2] L. Susskind, J. Math. Phys. (N.Y.) **36**, 6377 (1995).
[3] N. Goheer, M. Kleban, and L. Susskind, J. High Energy Phys. **07** (2003) 056.
[4] T. Banks and L. Susskind, Phys. Rev. D **54**, 1677 (1996).
[5] P. Martin and J. Schwinger, Phys. Rev. **115**, 1342 (1959); J. Schwinger, J. Math. Phys. (N.Y.) **2**, 407 (1961); K. T. Mahanthappa, Phys. Rev. **126**, 329 (1962); P. M. Bakshi and K. T. Mahanthappa, J. Math. Phys. (N.Y.) **4**, 1 (1963); **4**, 12 (1963); L.V. Keldysh, Zh. Eksp. Teor. Fiz. **47**, 1515 (1964) Sov. Phys. JETP **20**, 1018 (1965).Y. Takahashi and H. Umezawa, Collective Phenomena **2**, 55 (1975).
[6] W. Israel, Phys. Lett. **57A**, 107 (1976).
[7] J. M. Maldacena, J. High Energy Phys. **04** (2003) 021.
[8] L. Dyson, M. Kleban, and L. Susskind, J. High Energy Phys. **210** (2002) 011.