Perfect State Transfer in Quantum Spin Networks

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We propose a class of qubit networks that admit the perfect state transfer of any quantum state in a fixed period of time. Unlike many other schemes for quantum computation and communication, these networks do not require qubit couplings to be switched on and off. When restricted to N-qubit spin networks of identical qubit couplings, we show that $2\log_3 N$ is the maximal perfect communication distance for hypercube geometries. Moreover, if one allows fixed but different couplings between the qubits, then perfect state transfer can be achieved over arbitrarily long distances in a linear chain.

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The transfer of quantum states from one location (A) to another (B) is an important feature in many quantum information processing systems. Depending on the technology at hand, this task can be accomplished in a number of ways. Optical systems, typically employed in quantum communication and cryptography applications, transfer states from A to B directly via photons. These photons could contain an actual message or could be used to create entanglement between A and B for future quantum teleportation between the two sites [1]. Quantum computing applications with trapped atoms use a variety of information carriers to transfer states from A to B, e.g., photons in cavity QED [2] and phonons in ion traps [3]. These photons and phonons may be viewed as individual quantum carriers. However, many promising technologies for the implementation of quantum information processing, such as optical lattices [4], and arrays of quantum dots [5] rely on collective phenomena to transfer quantum states. In this case, a "quantum wire," the most fundamental unit of any quantum processing device, is made out of many interacting components. In the sequel, we focus on quantum channels of this type. Insight into the physics of perfect quantum channels is of special significance for technologies that route entanglement and quantum states on networks. These technologies range from the very small, such as the components of a quantum cellular automaton, to the medium sized, such as the data bus of a quantum computer, to the truly grand, such as a quantum Internet spanning many quantum computers.

In this Letter, we address the problem of arranging N interacting qubits in a network which allows the perfect transfer of any quantum state over the longest possible distance. The transfer is implemented by preparing the input qubit A in a prescribed quantum state and, some time later, by retrieving the state from the output qubit B. The network is described by a graph G in which the vertices $V(G)$ represent locations of the qubits and a set of edges $E(G)$ specifies which pairs of qubits are coupled. The graph is characterized by its adjacency matrix $A(G)$,

$$A_{ij}(G) := \begin{cases} 1 & \text{if } (i, j) \in E(G) \\ 0 & \text{otherwise.} \end{cases} \quad (1)$$

It has two special vertices, labeled A and B, which mark the input and the output qubits, respectively. We define the distance between A and B to be the number of edges constituting the shortest path between them. Although this distance is defined on a graph, it is directly related to the physical separation between the input and output qubits, when a graph can be embedded in physical space.

The most desirable graph, for our purposes, is a linear chain of N qubits with A and B at the two opposite ends of the chain. For fixed N it maximizes the distance between A and B. If we can switch on and off couplings between adjacent qubits, then we can swap qubit states one-by-one along the chain, all the way from the input to the output. Such a dynamical control over the interactions between the qubits is still an experimental challenge. Theoretically, this challenge has been met by considerable progress in reducing the amount of control needed to accomplish quantum computation tasks [6]. Moreover, it has been shown that this can also be achieved without direct control over interqubit interactions, as long as one has control over individual qubits [7]. Even if just one qubit in the chain is controllable, then quantum communication can be effected [8].

Quantum communication over short distances through an unmodulated spin chain has been studied in detail, and an expression for the fidelity of quantum state has been obtained [9]. In contrast to these works, we focus on the situation in which the state transfer is perfect; i.e., the fidelity is unity, and can be achieved over arbitrarily long distances.

We consider networks in which state transfer is achieved by time evolution under a suitable time-independent Hamiltonian, without any additional external control. This mechanism avoids possible errors arising from dynamical control of interqubit interactions. However, note that we do not consider the effects of any other source of errors in this Letter.

Here, we show that a simple XY coupling

$$H_G = \frac{1}{2} \sum_{(i,j)\in E(G)} [\sigma_i^x \sigma_j^x + \sigma_i^y \sigma_j^y], \tag{2}$$

where σ_i^x, σ_i^y, and σ_i^z are the Pauli matrices acting on the ith qubit, allows perfect state transfer between antipodes of a hypercube. Moreover, if we can engineer the strength of the couplings between the qubits then perfect state transfer can also be performed between the two ends of a linear chain. These tasks can also be accomplished with the Heisenberg, or exchange, interaction by a suitable modulation of the network, e.g., by placing it in a static but, in general, nonuniform external magnetic field. We shall elaborate on this point later on.

Although our qubits represent generic two state systems, for the convenience of exposition we will also use the term spin as it provides a simple physical picture of the network. The standard basis for an individual qubit is chosen to be $\{|0\rangle \equiv |\downarrow\rangle, |1\rangle \equiv |\uparrow\rangle\}$, and we shall assume that initially all spins point "down" along a prescribed z axis; i.e., the network is in the state $|\underline{0}\rangle = |0_A 00\cdots 00_B\rangle$. This is an eigenstate of the Hamiltonian (2) corresponding to zero energy.

The Hilbert space $\mathcal{H}_G$ associated with a network of N qubits is of dimension 2^N. However, the state transfer dynamics is completely determined by the evolution in the N-dimensional subspace S_G spanned by the basis vectors $|n\rangle$, $n = 1, \ldots, N$, corresponding to spin configurations in which all spins are down apart from just one spin at the vertex n which is "up." Indeed, when we prepare the input qubit A in state $\alpha|0\rangle + \beta|1\rangle$, the state of the network becomes

$$\alpha|0_A 00\cdots 00_B\rangle + \beta|1_A 00\cdots 00_B\rangle = \alpha|\underline{0}\rangle + \beta|1\rangle. \tag{3}$$

The coefficient α does not change in time, as $|\underline{0}\rangle$ is the zero-energy eigenstate of H_G. The operator of the total z component of the spin,

$$\sigma_{\text{tot}}^z := \sum_{i\in V(G)} \sigma_i^z, \tag{4}$$

commutes with H_G, which leads to the conservation of the total z component of spin. This means that the state $|1\rangle \equiv |1_A 00\cdots 00_B\rangle$ must evolve into a superposition of states with exactly one spin up and all other spins down. Thus, the initial state of the network evolves in time t as

$$\alpha|\underline{0}\rangle + \beta|1\rangle \mapsto \alpha|\underline{0}\rangle + \sum_{n=1}^{N} \beta_n(t)|n\rangle. \tag{5}$$

The dynamics are effectively confined to the subspace S_G. The Hamiltonian H_G, when restricted to this subspace, is represented by an $N \times N$ matrix that is *identical* to the adjacency matrix $A(G)$, Eq. (1), of the underlying graph G. Because of this, one may express the time evolution of the network in the S_G subspace as a continuous-time quantum walk on G (first considered by Farhi and Gutmann in 1998 [10]).

The question we are interested in is: When will the quantum walk propagate from A to B with unit fidelity? To answer this, if we identify qubit A with vertex 1 and qubit B with vertex N, we need to compute the probability amplitude that the network initially in state $|1\rangle$, corresponding to $|1_A 00\cdots 00_B\rangle$, evolves after time t to state $|N\rangle$, corresponding to $|0_A 00\cdots 01_B\rangle$; i.e.,

$$F(t) = \langle N|e^{-itH_G}|1\rangle. \tag{6}$$

Perfect state transfer is obtained for times t for which $|F(t)| = 1$. Here and henceforth we take $\hbar = 1$.

Let us start with the XY linear chain of qubits. In this case, one can compute $F(t)$ explicitly by diagonalizing the Hamiltonian or the corresponding adjacency matrix. The eigenstates are given by

$$|\tilde{k}\rangle = \sqrt{\frac{2}{N+1}} \sum_{n=1}^{N} \sin\left(\frac{\pi k n}{N+1}\right)|n\rangle, \tag{7}$$

with corresponding eigenvalues $E_k = -2\cos\frac{k\pi}{N+1}$ for all $k = 1, \ldots, N$. Thus,

$$F(t) = \frac{2}{N+1} \sum_{k=1}^{N} \sin\left(\frac{\pi k}{N+1}\right) \sin\left(\frac{\pi k N}{N+1}\right) e^{-iE_k t}. \tag{8}$$

Perfect state transfer from one end of the chain to another is possible *only* for $N = 2$ and $N = 3$, with $F(t) = -i\sin(t)$ and $F(t) = -[\sin(t/\sqrt{2})]^2$, respectively. For perfect state transfer in a chain, it is necessary that the ratios of the differences of eigenvalues of the related adjacency matrix $A(G)$ are rational numbers. The absence of perfect state transfer for $N \geq 4$ can be proven by showing explicitly that the above condition is not satisfied.

A chain of two or three qubits can serve as basic building blocks for networks that can perfectly transfer a quantum state over longer distances. This can be achieved by building networks which are multiple Cartesian products of either of the two simple chains.

In general, the Cartesian product of two graphs $G := \{V(G), E(G)\}$ and $H := \{V(H), E(H)\}$ is a graph $G \times H$ whose vertex set is $V(G) \times V(H)$ and two of its vertices (g, h) and (g', h') are adjacent if and only if one of the following hold: (i) $g = g'$ and $\{h, h'\} \in E(H)$; (ii) $h = h'$ and $\{g, g'\} \in E(G)$. If $|\tilde{k}\rangle$ is an eigenvector of $A(G)$ corresponding to eigenvalue E_k and $|\tilde{l}\rangle$ is an eigenvector of $A(H)$ corresponding to eigenvalue E_l, then $|\tilde{k}\rangle \otimes |\tilde{l}\rangle$ is an eigenvector of $A(G \times H)$ corresponding to eigenvalue

$E_k + E_l$. This is because

$$A(G \times H) = A(G) \otimes \mathbb{1}_{V(G)} + \mathbb{1}_{V(H)} \otimes A(H), \qquad (9)$$

where $\mathbb{1}_{V(H)}$ is the $|V(H)| \times |V(H)|$ identity matrix (see, e.g., [11]).

Now, consider a graph G^d which is a d-fold Cartesian product of graph G. The propagator between the two antipodal vertices in G^d, namely, $A = (1, \ldots, 1)$ and $B = (N, \ldots, N)$, is simply

$$F_{G^d}(t) = [F_G(t)]^d. \qquad (10)$$

The d-fold Cartesian product of a one-link chain (two qubits) and a two-link chain (three qubits) lead to one-link and two-link hypercubes with $|F(t)|$ given, respectively, by

$$[\sin(t)]^d \quad \text{and} \quad \left[\sin\left(\frac{t}{\sqrt{2}}\right)\right]^{2d}. \qquad (11)$$

Any quantum state can be perfectly transferred between the two antipodes of the one-link and two-link hypercubes of any dimensions in constant time $t = \pi/2$ and $t = \pi/\sqrt{2}$, respectively [12].

Let us mention in passing that our discussion here is related to comparative studies of continuous-time random walks on graphs. The mean hitting time between vertices A and B is the time it takes the random walk on average to reach B starting at A. The classical mean hitting time between the antipodes in a one-link and two-link d-dimensional hypercube is given, for large d, by 2^d and 3^d, respectively. One way to calculate this is to reduce the continuous-time random walk on the d-dimensional hypercube G^d to a continuous-time random walk, with potential drift, in *one* and *two* dimensions, respectively, via the so-called lumping method (see, e.g., [13]). In contrast, as we have shown, the corresponding quantum hitting time is constant. This gives an exponential separation between the classical and quantum mean hitting times between the antipodes of one- and two-link hypercubes.

Thus, we have shown that, for a two-link hypercube of N sites, the maximum distance of perfect quantum communication is $2\log_3 N$. It is an interesting open problem to see if, given N qubits, one can construct a network with identical couplings in which any quantum state can be perfectly transferred over a larger distance.

An improvement of the perfect quantum communication distance to N is, however, possible if one allows for different, but fixed, couplings between qubits on a chain. In order to see how to do this, let us start with a convenient relabeling of qubits. One may associate a fictitious spin-$(N-1)/2$ particle with an N-qubit chain and relabel the basis vectors as $|m\rangle$, where $m = -\frac{1}{2}(N-1) + n - 1$.

The input node $|A\rangle$ can be labeled both as $|n = 1\rangle$ and $|m = -\frac{1}{2}(N-1)\rangle$, and the output node $|B\rangle$ both as $|n = N\rangle$ and $|m = +\frac{1}{2}(N-1)\rangle$. An example for $N = 6$ is depicted in Fig. 1. Now let the evolution of the chain be governed by a modified version of (2),

$$H_G = \sum_{(n,n+1)\in E(G)} \frac{J_n}{2}[\sigma_n^x \sigma_{n+1}^x + \sigma_n^y \sigma_{n+1}^y], \qquad (12)$$

which, when restricted to the subspace S_G, is of the form

$$\begin{pmatrix} 0 & J_1 & 0 & \cdots & 0 \\ J_1 & 0 & J_2 & \cdots & 0 \\ 0 & J_2 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & J_{N-1} \\ 0 & 0 & 0 & J_{N-1} & 0 \end{pmatrix}. \qquad (13)$$

The above matrix, Eq. (13), is identical to the representation of the Hamiltonian H of a fictitious spin $S = \frac{1}{2}(N-1)$ particle: $H = \lambda S_x$, where S_x is its angular momentum operator and λ is some constant. In this case, the matrix elements J_n are equal to $\frac{\lambda}{2}\sqrt{n(N-n)}$. The evolution

$$U(t) = \exp(-i\lambda t S_x), \qquad (14)$$

of the network represents a rotation of this fictitious particle. The matrix elements $\langle n'|U(t)|n\rangle$ of this rotation matrix are well known and, in particular, the probability amplitude for state transfer is

$$F(t) = \langle N|U(t)|1\rangle = \left[-i\sin\left(\frac{\lambda t}{2}\right)\right]^{N-1}. \qquad (15)$$

Thus, perfect transfer of a quantum state between the two antipodes A and B is obtained in a constant time $t = \pi/\lambda$.

Each such engineered qubit chain can be viewed as a projection from a graph having identical qubit couplings. In fact, there is an entire family of such graphs $\mathcal{G}$ that project to this chain. Motivated by the "column method" of [14], we define $\mathcal{G}$ as the set of graphs whose vertices can be partitioned into N columns G_n of size $|G_n| = \binom{N-1}{n-1}$ that satisfy the following two conditions for $n = 1, \ldots, N$: (i) each vertex in column n is connected to $N - n$ vertices in column $n+1$, and (ii) each vertex in column $n+1$ is connected to n vertices in column n. An important example of a graph in $\mathcal{G}$ is the one-link hypercube,

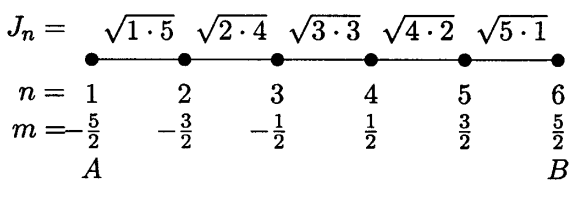

FIG. 1. Couplings J_n that admit perfect state transfer from A to B in a six-qubit chain. Eigenvalues m of the equivalent spin-$\frac{5}{2}$ particle are also shown.

where columns are defined as the set of vertices reachable in n links. The evolution of a state at A (the first column) under H_G [Eq. (2)] remains in the column space $\mathcal{H}_{\text{col}} \subseteq \mathcal{H}_G$, spanned by

$$|\text{col } n\rangle = \frac{1}{\sqrt{|G_n|}} \sum_{m=1}^{|G_n|} |G_{n,m}\rangle \tag{16}$$

where $G_{n,m}$ labels the vertices in G_n. Hence, we restrict our attention to $\mathcal{H}_{\text{col}}$ in which the matrix elements of H_G are given by

$$J_n = \langle \text{col } n | H_G | \text{col } n+1 \rangle = \sqrt{n(N-n)}, \tag{17}$$

the same as in the engineered chain.

In our analysis, we have focused on qubits coupled with the XY interaction. The choice of this interaction was dictated by its simple connection with the adjacency matrix. We should add, however, that our considerations remain valid if we choose the Heisenberg interaction and compensate for the diagonal elements in the S_G subspace. For example, the Heisenberg model with local magnetic fields,

$$\frac{1}{2} \sum_{j=1}^{N-1} J_j \vec{\sigma}_j \cdot \vec{\sigma}_{j+1} + \sum_{j=1}^{N} B_j \sigma_j^z, \tag{18}$$

where $\vec{\sigma}_j = (\sigma_j^x, \sigma_j^y, \sigma_j^x)$ and $B_n = \frac{1}{2}(J_{n-1} + J_n) - \frac{1}{2(N-2)} \sum_{k=1}^{N-1} J_k$, give exactly the same state transfer dynamics as the XY model.

Our analysis is not restricted to pure states; the method presented here works equally well for mixed states. It can also be used to transfer or to distribute quantum entanglement.

In conclusion, in this Letter we have proven that perfect quantum state transfer between antipodal points of one-link and two-link hypercubes is possible, and perfect quantum state transfer between antipodal points of N-link hypercubes for $N \geq 3$ is impossible. The transfer time on these hypercubes is independent of their dimension. In addition, we have shown that a quantum state can be transferred perfectly over a chain of *any* length as long as one can pre-engineer interqubit interactions. These networks are especially appealing as they require no dynamical control, unlike many other quantum communication proposals.

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