Topology and H-Flux of T-Dual Manifolds

Peter Bouwknegt,[1,2] Jarah Evslin,[3] and Varghese Mathai[2]
[1]Department of Physics, University of Adelaide, Adelaide, SA 5005, Australia
[2]Department of Pure Mathematics, University of Adelaide, Adelaide, SA 5005, Australia
[3]INFN Sezione di Pisa, Via Buonarroti, 2, Edificio C, 56127 Pisa, Italy
(Received 1 January 2004; published 6 May 2004)

We present a general formula for the topology and H-flux of the T-dual of a type II compactification. Our results apply to T-dualities with respect to any free circle action. In particular, we find that the manifolds on each side of the duality are circle bundles whose curvatures are given by the integral of the dual H-flux over the dual circle. As a corollary we conjecture an obstruction to multiple T-dualities, generalizing the obstruction known to exist on the twisted torus. Examples include SU(2) Wess-Zumino-Witten models, lens spaces, and the supersymmetric string theory on the nonspin $AdS^5 \times \mathbb{C}P^2 \times S^1$ compactification

DOI: 10.1103/PhysRevLett.92.181601 PACS numbers: 11.25.Tq, 11.25.Hf, 11.25.Mj

T-duality is a generalization of the $R \rightarrow 1/R$ invariance of string theory compactified on a circle of radius R. The local transformation rules of the low energy effective fields under T-duality, known as the Buscher rules [1], have been known for some time. However, in cases in which there is a topologically nontrivial Neveu-Schwarz (NS) 3-form H-flux the Buscher rules make sense only on each local spacetime patch. Several examples of T-duals to such backgrounds have been found [2–5] and in each case it was seen that T-duality changes not only the H-flux, but also the spacetime topology. While any new set of equivalences between compactifications may be useful for model building, T-dualities involving H-flux have also led to the surprising discovery of the new physical phenomenon "supersymmetry without supersymmetry" in Refs. [3,6].

In this Letter, which is a physicist's perspective of Ref. [7], we present a general formula for the topology and H-flux of a compactification from the topology and H-flux of its T-dual. As supporting evidence, we have shown [7] that locally our formula agrees with the Buscher rules and that globally it yields an isomorphism of the twisted K-theory-valued conserved Ramond-Ramond (RR) charges [8–10]. Here we demonstrate that our formula reproduces the old examples of topology change and easily generates many more. Proofs, physical motivations, more complete referencing, and several applications (including some tantalizingly mysterious clues to F-theory) may be found in the companion paper, Ref. [7].

Henceforth we restrict our attention to type IIA and type IIB string theories on the 10-manifolds E and $\hat{E}$, respectively, which admit the free circle actions

$$f : S^1 \times E \rightarrow E, \qquad \hat{f} : \hat{S}^1 \times \hat{E} \rightarrow \hat{E}. \tag{1}$$

The spaces of orbits of these actions are 9-manifolds which we call M and $\hat{M}$. The freeness of the actions implies that each orbit is a loop and that none of these loops degenerate. As a result E and $\hat{E}$ are circle bundles over the bases M and $\hat{M}$, and so their topologies are entirely determined by the topology of their bases together with the curvatures F, $\hat{F}$ (the first Chern classes of the bundles).

As we will see nontrivial bundles are T-dual to configurations with H-flux. Thus we will need to include the fluxes H and $\hat{H}$ in our two compactifications. The two configurations are then topologically determined by the triples (M, F, H) and $(\hat{M}, \hat{F}, \hat{H})$ where M and $\hat{M}$ are 9-manifolds, F and $\hat{F}$ are two-forms on M and $\hat{M}$, and H and $\hat{H}$ are three-forms on the total spaces E and $\hat{E}$. To capture the topology of a configuration it will suffice to consider the field strengths and Chern classes as elements of integral cohomology. This perspective is useful in that it automatically identifies some gauge equivalent configurations, excludes configurations not satisfying some equations of motion, and imposes the Dirac quantization conditions.

Our result.—The compactifications topologically specified by (M, F, H) and $(\hat{M}, \hat{F}, \hat{H})$ are T-dual if and only if

$$M \cong \hat{M}, \qquad F = \int_{\hat{S}^1} \hat{H}, \qquad \hat{F} = \int_{S^1} H. \tag{2}$$

This condition determines, at the level of cohomology, the curvatures F and $\hat{F}$. However, the NS field strengths are determined only up to the addition of a three-form on the base M, because the integral of such a form over the circle fiber vanishes. We further impose that the two dual three-forms on M must be equal, as is made precise in Ref. [7], where the duality map of the RR fields (viewed both as elements of cohomology and of K-theory) can also be found.

In the remainder of this Letter we provide examples and applications of our result. When the curvatures F and $\hat{F}$ are topologically trivial (in the second cohomology of M) the bundles are trivial and so our two spacetimes are both topologically the trivial bundle $M \times S^1$. Using the Künneth formula we may decompose the third cohomology of the total space $M \times S^1$,

$$H^3(M \times S^1) = H^3(M) \otimes H^0(S^1) \oplus H^2(M) \otimes H^1(S^1) = H^3(M) \oplus H^2(M), \quad (3)$$

and so the NS fluxes H and $\hat{H}$, being elements of $H^3(M \times S^1)$, decompose as $H = \alpha + \beta d\theta$, $\hat{H} = \hat{\alpha} + \hat{\beta} d\theta$, where $\alpha, \hat{\alpha} \in H^3(M)$, $\beta, \hat{\beta} \in H^2(M)$, and $d\theta$ is the generator of $H^1(S^1) = \mathbb{Z}$. Integrating H and $\hat{H}$ over the circle, using the normalization $\int d\theta = 1$, our result yields $\alpha = \hat{\alpha}$, $\beta = \hat{\beta} = 0$. Thus we reproduce the original examples of T-duality, in which spacetime is the product of a 9-manifold and a circle and the H-flux is an element of the cohomology of the 9-manifold. As expected the T-dual is also a product manifold and carries the same H-flux.

The next most trivial case is a trivial circle bundle with H-flux, which we see from (2) is T-dual to a nontrivial bundle without H-flux. In this case our result was demonstrated using S-duality and also using the E_8 gauge bundle formalism in Ref. [7].

The simplest nontrivial circle bundle is the Hopf fibration over the 2-sphere. This bundle is constructed by cutting the 2-sphere into a northern S^2_N and southern hemisphere S^2_S, over which the circle is trivially fibered. The two hemispheres are then glued together along the equator. In particular, each point on the equator is specified by a longitude θ and the attaching map identifies the point ϕ on the fiber over one hemisphere to the point $\phi + \theta$ on the other. The total space of this bundle is then the three-sphere S^3 and the curvature F is the generator [1] of $H^2(S^2) = \mathbb{Z}$.

Now we may consider a type II string theory compactification on a 3-sphere crossed with an irrelevant 7-manifold, express the 3-sphere as the circle bundle above, and then T-dualize with respect to the circle fiber.

Let us begin with a case in which there is no H-flux, such as a type IIB compactification of $\mathrm{AdS}^3 \times T^4 \times S^3$ supported by RR flux. Applying Eq. (2) with $F = [1] \in H^2(S^2)$ and $H = 0$, we find $\hat{F} = [0]$ and $\hat{H} = [1] \in H^3(S^2 \times S^1)$. Thus our nontrivial bundle S^3 becomes the trivial bundle $S^2 \times S^1$, supported by one unit of H-flux. Incidentally the construction in Ref. [7] tells us that if there was RR 3-form flux $G_3 = [k] \in H^3(S^3)$ on the S^3, then we would find $\hat{G}_2 = [k] \in H^2(S^2)$, whereas the G_3-flux on the AdS^3 becomes $\hat{G}_4$-flux on $\mathrm{AdS}^3 \times S^1$. This is in accord with the usual intuition in which T-duality toggles whether an RR flux extends along the circle, except that here we see that this intuition may apply even when the circle is nontrivially fibered. One might think that this would be impossible in general because, for example, here, there is no first cohomology class corresponding to the circle in the 3-sphere. This is potentially problematic because, for example, an arbitrary integral Romans mass $\hat{G}_0 \in H^0(S^2 \times S^1)$ cannot be dual to an element $G_1 \in H^1(S^3)$ since $H^1(S^3) = 0$. Here we are saved from any contradiction by the supergravity equation of motion $\hat{G}_0 H = 0$, which implies that $\hat{G}_0 = [0]$. More generally we are saved by the quantum corrected equations of motion, which are given by the Freed-Witten anomaly [11].

A famous application of the previous example is the T-duality of string theory on $\mathbb{R}^{8,1} \times S^1$ with an NS5-brane extended along the plane $\mathbb{R}^{5,1} \subset \mathbb{R}^{8,1}$ and localized at a point $\theta \in S^1$. Such an NS5-brane is linked by an $S^2 \times S^1$ where $S^2 \subset \mathbb{R}^{8,1}$ links the $\mathbb{R}^{5,1}$ plane. Recalling that NS5-branes are magnetic sources for NS flux, Gauss's law allows us to integrate $\int_{S^2 \times S^1} H = 1$ and so

$$H = [1] \in H^3(S^2 \times S^1) = \mathbb{Z}. \quad (4)$$

As we have seen, a T-duality along this circle replaces $S^2 \times S^1$ with a 3-sphere and the H-flux disappears. As the H-flux has disappeared the T-dual compactification has no NS5-brane; instead it has been replaced by a circle bundle which is nontrivially fibered over each sphere linking the 6-submanifold where the NS5-brane was. This submanifold is now a Kaluza-Klein (KK) monopole for the dual U(1). Thus we have recovered the familiar fact that NS5-branes are T-dual to KK monopoles (see, e.g., [12], and references therein). If, instead of a single NS5-brane, we had considered a stack of k NS5-branes, then we would have had $H = [k]$ and so the dual bundle would again have been nontrivial, this time yielding a KK-monopole charge of k. More generally we may consider nontrivial circle bundles and nontrivial H-flux at the same time. For example, string theory on the Lens space $S^3/\mathbb{Z}_j$ with k units of H-flux is T-dual to string theory on $S^3/\mathbb{Z}_k$ with j units of H-flux [13].

An example of T-duality that has recently received attention in the literature [5] is the duality between circle bundles over a 2-torus T^2. In particular, one may start with a 3-torus, which is the trivial circle bundle over a 2-torus, with $k \neq 0$ units of H-flux and then T-dualize with respect to the circle fiber. As above, $F = [0] \in H^2(T^2) = \mathbb{Z}$ and $H = [k] \in H^3(T^3) = \mathbb{Z}$ determine the dual curvature and NS flux

$$\hat{F} = [k] \in H^2(T^2) = \mathbb{Z}, \qquad \hat{H} = [0] \in H^3(\hat{E}) = \mathbb{Z}. \quad (5)$$

Here the dual manifold $\hat{E}$, commonly referred to as a twisted torus or nilmanifold, is the circle bundle over T^2 characterized entirely by the curvature $\hat{F} = [k]$.

We may also try to T-dualize a larger subtorus of the original T^3. This means that after T-dualizing with respect to the fiber circle we may then try to T-dualize with respect to one of the circles in the base. As has been found in Ref. [5], this is impossible. In particular, after the first circle is T-dualized, the other two circles have ceased to be globally defined and so cannot be T-dualized. Had they both been globally defined we could have defined a nowhere-vanishing section of this nontrivial circle bundle. Thus in general we see that it is impossible to T-dualize with respect to any 3-torus supporting H-flux. In fact, the example in Ref. [5] suggests the stronger result that one cannot T-dualize on a 2-torus, unless $\int_{T^2} H = 0$ in cohomology.

A critical check of any proposed duality is that the anomalies match on both sides. Although a more general matching of a particular gravitino anomaly was demonstrated in Ref. [7], here we describe one family of examples which illustrates the general pattern. Consider the famous type IIB string theory compactification on $\mathrm{AdS}^5 \times S^5$, where AdS^5 is the five-dimensional anti–de-Sitter space. The 5-sphere S^5 is a circle bundle over the complex projective plane $\mathbb{C}\mathrm{P}^2$ with a single unit of curvature $F = [1] \in H^2(\mathrm{AdS}^5 \times \mathbb{C}\mathrm{P}^2) = H^2(\mathbb{C}\mathrm{P}^2) = \mathbb{Z}$. The second equality is a result of the fact that AdS^5 is contractible and so it does not contribute to the cohomology groups; we may thus freely omit it from such equations. The third cohomology group of this spacetime is trivial and so the H-flux must be topologically trivial. In addition, there is a RR 5-form field strength $G_5 = [N] \in H^5(S^5)$. There is also G_5-flux along the noncompact directions.

T-dualizing with respect to the fiber circle we find that the dual curvature $\hat{F}$ vanishes. The dual manifold is thus $\mathrm{AdS}^5 \times \mathbb{C}\mathrm{P}^2 \times S^1$. Equation (2) is satisfied only if the dual NS field strength is

$$\hat{H} = [1] \in H^3(\mathrm{AdS}^5 \times \mathbb{C}\mathrm{P}^2 \times S^1) = H^2(\mathbb{C}\mathrm{P}^2) \otimes H^1(S^1) = \mathbb{Z} \otimes \mathbb{Z} = \mathbb{Z}. \tag{6}$$

The G_5-flux becomes $\hat{G}_4 = [N] \in H^4(\mathbb{C}\mathrm{P}^2) = \mathbb{Z}$ and $\hat{G}_6$ is supported in the noncompact directions crossed with the circle. In particular, there is no $\hat{G}_2$-flux and therefore the M-theory circle is trivially fibered, meaning that this configuration is dual to M-theory compactified on $\mathrm{AdS}^5 \times \mathbb{C}\mathrm{P}^2 \times T^2$. Notice that $\mathbb{C}\mathrm{P}^2$ is not *spin*, and so this is an M-theory compactification on a *nonspin* manifold. This might seem impossible, because the low energy effective supergravity has a gravitino and so there must be a spin structure. But, in fact, this compactification was seen to be perfectly consistent and even supersymmetric (although the supergravity truncation is not supersymmetric, as the winding modes are required to complete the supermultiplets) in Ref. [3]. To see this, notice that the potential gravitino anomaly should be visible in the low energy theory, in which we throw away KK modes along the torus. This leaves an effective 9-dimensional theory in which the gravitino is electrically charged under a U(1) gauge field strength which is the integral of H over the dual circle. By Eq. (2) this integral is just the original curvature F, so we see that our result implies that the original (IIB) circle is the U(1) gauge bundle under which the gravitino is charged. This fact would have been obvious if we had dimensionally reduced the IIB side instead. When F is odd the Skyrme effect then assures that the gravitino is, in fact, a boson and so its phase is well defined even on a spacetime which is not spin. In our case $F = [1]$ and so the compactification is consistent.

On the other hand, in the case $F = [2]$ the gravitino still behaves as a fermion and so the IIA configuration is inconsistent. In this case the IIB spacetime is $\mathrm{AdS}^5 \times \mathbb{R}\mathrm{P}^5$, where $\mathbb{R}\mathrm{P}^5$ is the five-dimensional real projective space, which is an inconsistent background in the absence of an orientifold plane. Our result applies only to oriented theories, and, in particular, this IIB compactification is inconsistent in an oriented theory, in agreement with the inconsistency of the IIA side of the duality. In fact, the total anomalies on both sides always agree: it seems as though both sides are free from this anomaly when the F-theory 12-manifold is spin [7]. This connection to F-theory is tantalizing and mysterious.

The Roček-Verlinde [14] approach to T-duality should be applicable to circle bundles with nontrivial H-flux as well, hopefully reproducing our result directly within the nonlinear sigma model context. In fact, such an approach has already been attempted in Ref. [2], but this program may be significantly easier with a proposal already at hand. We hope that such an approach would also shed light on the observed obstruction to T-dualizing 2-subtorii that support a topologically nontrivial H-flux. A more difficult problem is to generalize our results to nonfree circle actions [15]; this may yield a formula for the topology of mirror manifolds.

We thank Nick Halmagyi, Michael Schulz, and Eric Sharpe for help and crucial references. P. B. and V. M. are financially supported by the Australian Research Council, and J. E. by the INFN Sezione di Pisa.

[1] T. Buscher, Phys. Lett. B **194**, 59 (1987); **201**, 466 (1988).
[2] E. Álvarez, L. Álvarez-Gaumé, J. L. F. Barbón, and Y. Lozano, Nucl. Phys. **B415**, 71 (1994).
[3] M. J. Duff, H. Lü, and C. N. Pope, Nucl. Phys. **B532**, 181 (1998).
[4] S. Gurrieri, J. Louis, A. Micu, and D. Waldram, Nucl. Phys. **B654**, 61 (2003).
[5] S. Kachru, M. Schulz, P. Tripathy, and S. Trivedi, J. High Energy Phys. 03 (2003) 061.
[6] I. Bakas, Phys. Lett. B **343**, 103 (1995); I. Bakas and K. Sfetsos, Phys. Lett. B **349**, 448 (1995); E. Álvarez, L. Álvarez-Gaumé, and I. Bakas, Nucl. Phys. **B457**, 3 (1995).
[7] P. Bouwknegt, J. Evslin, and V. Mathai, hep-th/0306062 [Commun. Math. Phys. (to be published)].
[8] R. Minasian and G. Moore, J. High Energy Phys. 11 (1997) 002.
[9] E. Witten, J. High Energy Phys. 12 (1998) 019.
[10] P. Bouwknegt and V. Mathai, J. High Energy Phys. 03 (2000) 007.
[11] D. Freed and E. Witten, hep-th/9907189.
[12] D. Tong, J. High Energy Phys. 07 (2002) 013.
[13] J. Maldacena, G. Moore, and N. Seiberg, J. High Energy Phys. 10 (2001) 005.
[14] M. Roček and E. Verlinde, Nucl. Phys. **B373**, 630 (1992).
[15] R. Donagi and T. Pantev, math.AG/0306213.