Stability of Macroscopic Entanglement under Decoherence

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We investigate the lifetime of macroscopic entanglement under the influence of decoherence. For Greenberger-Horne-Zeilinger–type superposition states, we find that the lifetime decreases with the size of the system (i.e., the number of independent degrees of freedom), and the effective number of subsystems that remain entangled decreases with time. For a class of other states (e.g., cluster states), however, we show that the lifetime of entanglement is independent of the size of the system.

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The question as to whether or not entanglement—a genuine feature of quantum mechanics—can persist in a macroscopic (i.e., "classical") world has entertained quantum physicists since Schrödinger introduced his notorious gedanken experiment known as "Schrödinger's cat" [1,2]. While entangled states of microscopic matter, such as a few atoms or ions in a trap, can now be prepared in the laboratory [3], it is often argued that for a large number N of particles this task would become exceedingly difficult, since the effective decoherence rate would grow linearly with the size of the system N.

In its simplest version, the argument is based on the observed evolution of superposition states of the form $|\mathrm{GHZ}\rangle \equiv 1/\sqrt{2}(|0\rangle^{\otimes N} + |1\rangle^{\otimes N})$, also called Greenberger-Horne-Zeilinger (GHZ) states, of a system of N spins or qubits interacting with uncontrollable degrees of freedom of the environment, described, e.g., by a heatbath. The rate at which this state decoheres scales indeed as κN, where κ is the decoherence rate of a single qubit. While this observation is correct, it is not clear whether the scaling of the decoherence rate with the size of the system N is a special property of GHZ states or a general feature of all multiparticle entangled states. This means that the conclusions that are usually drawn from this observation, namely, that macroscopic entanglement, i.e., entanglement between a macroscopic number of particles, necessarily becomes exponentially fragile with N, are questionable and will indeed be refuted in this Letter.

To this aim, we investigate the effect of decoherence on the entanglement properties of a class of multiparticle entangled states. We consider the lifetime of entanglement between a variable number of subsystems of the system. Specifically, we consider both the time after which the distillable entanglement of the state vanishes and the time after which the state becomes separable. These lifetimes are, in general, finite. We determine the scaling behavior of these lifetimes with N. For GHZ states we find that an increasing decoherence rate indeed results in a lifetime of distillable N-party entanglement [4,5] that decreases with N. For many other states (e.g., cluster states [6]), in contrast, we show that the lifetime of genuine multiparty entanglement is *independent* of the size of the system N. More specifically, we find that the lifetime of any state that belongs to the class of graph states $|\phi\rangle_G$ [6], which contains the GHZ and cluster state as particular cases, is bounded from below by a quantity which depends only on the maximum degree of the associated graph. We remark that for cluster states the degree of the graph is constant, while for GHZ states it increases with N [see Fig. 1(a)]. This implies that genuine multiparticle entanglement of a macroscopic number of particles is possible and can persist for time scales that are independent of the size of the system.

When describing multiparticle entanglement, a central notion is the *partitioning* of a system of N particles into $M \leq N$ groups. Each group may consist of several particles, which are then considered as a single subsystem with a higher-dimensional state space. If we associate a specific spatial distribution with the particles, as in the case of spins on a lattice, partitionings may be chosen that correspond to a rescaling of the size of the subsystems, as it is used in statistical physics [see Fig. 1(a)]. In the present context, we will be interested in the behavior of the distillable entanglement under a coarsening of the partitions. We will consider the case where the entire lattice of N qubits is in a graph state and compare, in particular, the two cases of GHZ states and cluster states. On the one

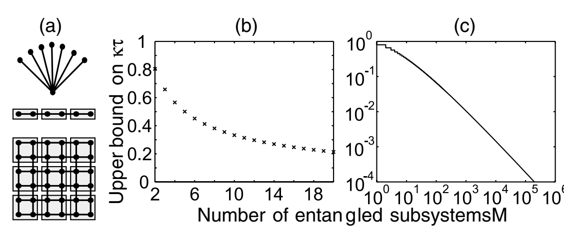

FIG. 1. (a) GHZ states, 1D and 2D cluster states. (b) Upper bound on lifetime $\kappa\tau$ of M-party entanglement for GHZ states in systems with $N \to \infty$ particles for different M. (c) Same as (b) but with double-logarithmic axis. Note that the same figures are obtained for the lifetime of N-particle entanglement in N-particle systems (see text).

hand, this approach allows us to determine the effective size of the system, i.e., the number of subsystems which are still entangled after a certain time. On the other hand, we can investigate the behavior of entanglement under rescaling in the asymptotic limit $N \to \infty$. We find that the lifetime of the distillable entanglement of the cluster state is largely independent of the size of the partitions, and thus the same on all scales. For the GHZ state, in contrast, we find that the distillable entanglement vanishes after an arbitrary short time on all scales, as long as we consider partitions of finite size. However, if we allow the sizes of the partitions to become macroscopic themselves (in the sense that the N qubits are divided into a fixed number of M cells whose size N/M grows to infinity as $N \to \infty$), then the lifetime of this M-party distillable entanglement becomes finite and scales to leading order as $1/(\kappa M)$.

Throughout this Letter we will mainly use a decoherence model corresponding to individual coupling of particles to a thermal bath in the large T limit, described by individual depolarizing channels. The same results can also be obtained, more generally, for models described by any quantum optical master equation of the Lindbladt form [7]. Furthermore, the results also hold for correlated noise with a finite correlation length.

We consider depolarizing channels with noise parameter $p \equiv p(t) \equiv e^{-\kappa t}$, where κ is a decay constant determined by the strength of the coupling to the environment and t is the interaction time. The channel acting on particle k is described by the completely positive map $\mathcal{E}_k \rho = p(t)\rho + \{[1 - p(t)]/4\}\sum_{j=0}^{3} \sigma_j^{(k)} \rho \sigma_j^{(k)}$, where σ_j are the Pauli matrices with $\sigma_0 \equiv \mathbb{1}$. We are interested in the entanglement of a given initially pure state $|\Psi\rangle$ as a function of time. That is, the initial state suffers from decoherence described by this model and evolves after time t to

$$\rho(t) = \mathcal{E}_1 \mathcal{E}_2 \cdots \mathcal{E}_N |\Psi\rangle\langle\Psi|. \tag{1}$$

We will use as a criterion the *distillable entanglement* of the system, which tells one whether it is possible to create true (irreducible) multiparticle entangled pure states. That is, we consider N distinct parties each holding a particle belonging to the N-particle state ρ. The state ρ is called N-party distillable entangled if there exists a local protocol (i.e., the parties act independently on their systems and are allowed to communicate classically) such that one can obtain from a sufficiently large number of copies of ρ some true N-particle entangled pure state [5]. Note that it is not necessary to specify the kind of multiparticle entangled pure state which is created, as all true N-party entangled pure states can be obtained from each other if several copies of the state are available [5]. We remark that a necessary condition for N-party distillability is that the partial transpositions with respect to any group of parties are nonpositive (see [4]).

We start out by investigating the entanglement properties of GHZ states under this decoherence model. One readily finds [8] that $|\mathrm{GHZ}\rangle$ evolves to a state $\rho(t)$ diagonal in the GHZ basis $\{|\Psi^{\pm}_{k_1 k_2 \cdots k_{N-1}}\rangle = 1/\sqrt{2}(|k_1 k_2 \cdots k_{N-1} 0\rangle \pm |\bar{k}_1 \bar{k}_2 \cdots \bar{k}_{N-1} 1\rangle)\}$, $k_j \in \{0, 1\}$, $\bar{k}_j = 1 - k_j$, with coefficients $\lambda_k^{\pm}$ that depend on $k \equiv \sum_j k_j$. For $k \neq 0$ we have $\lambda_k^+ = \lambda_k^- \equiv \lambda_k$, where $\lambda_k = [(1+p)^k (1-p)^{N-k} + (1+p)^{N-k}(1-p)^k]/2^{N+1}$, while $\lambda_0^+ = \lambda_0 + p^N/2$, $\lambda_0^- = \lambda_0 - p^N/2$. Note that $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_{[N/2]}$. The partial transposition with respect to a group B_k which contains exactly k parties is positive, $\rho(t)^{T_{B_k}} \geq 0$ if and only if [4] $p^N \geq 2\lambda_k$. For $k = 1$ one observes that the threshold value $p_{\mathrm{crit}} = e^{-\kappa t_{\mathrm{crit}}}$ where the partial transposition with respect to one particle becomes positive increases with N. This implies that for $t \geq t_{\mathrm{crit}} \equiv \tau$ the state is no longer N-party distillable entangled and thus the lifetime τ of true N-party entanglement decreases with the size of the system as expected [see Figs. 1(b) and 1(c)].

We now turn our attention to a larger class of multiparticle entangled states, the so-called graph states [6]. This class includes a variety of entangled states, e.g., GHZ states, cluster states, and code words of error correction codes [9]. Consider a graph $G = (V, E)$ which is a set of N vertices V connected in a specific way by edges E. The edges specify the neighborhood relation between vertices, and the degree of a vertex is given by the number of its neighbors. Graph states are associated with the "interaction" graph G as follows: Starting with a specific product state, the graph state $|\Psi_{\vec{0}}\rangle$ is obtained by applying an Ising-type interaction for time π between all neighboring vertices, i.e., $|\Psi_{\vec{0}}\rangle = \prod_{(k,l)\in E} U_{kl} |+\rangle^{\otimes N}$, where $U_{kl} \equiv e^{-i\pi[\mathbb{1}^{(k)} - \sigma_z^{(k)}]/2 \otimes (\mathbb{1}^{(l)} - \sigma_z^{(l)}]/2}$ and $|+\rangle = 1/\sqrt{2}(|0\rangle + |1\rangle)$. Equivalently [6], graph states can be described as joint eigenstates of a set of N commuting correlation operators K_j given by $K_j = \sigma_x^{(j)} \prod_{\{k,j\}\in E} \sigma_z^{(k)}$. The graph states $|\Psi_{\mu_1 \mu_2 \cdots \mu_N}\rangle$ fulfill the set of eigenvalue equations $K_j |\Psi_{\mu_1 \mu_2 \cdots \mu_N}\rangle = (-1)^{\mu_j} |\Psi_{\mu_1 \mu_2 \cdots \mu_N}\rangle \ \forall\ j$, $\mu_j \in \{0, 1\}$ and form a basis in $\mathcal{H} = (\mathbb{C}^2)^{\otimes N}$.

In the following, we consider a linear cluster state of N qubits [6] specified by the graph G with edges $\{k, k+1\}\ \forall\ k$ and investigate the entanglement properties of this state under the decoherence model described by Eq. (1). We show that the lifetime of distillable N-particle entanglement is for this state independent of the size of the system. In particular, we establish a lower bound $t_<$ on the lifetime t_{crit} of distillable N-party entanglement for such states which is independent of the number of particles N. This is in sharp contrast to the behavior of GHZ states.

In order to distill N-party entangled states, it is sufficient that maximally entangled pairs between all neighboring parties can be created, which we will show in the following. We emphasize that we use the distillability of neighboring pairs only as a tool to prove N-party distillability and it does not mean that the entanglement of the cluster state was in some sense only "bipartite." We make use of the following properties of graph states:

(i) Measuring all but two neighboring particles, say k, j, of a graph state $|\Psi_{\vec{0}}\rangle$ in the eigenbasis of σ_z results in the creation of another graph state with only a single edge $\{k, j\}$ [10]. The resulting state of particles k, j is up to local σ_z operations equivalent to a maximally entangled state of the form $|\Phi\rangle \equiv 1/\sqrt{2}(|0\rangle_x|0\rangle_z + |1\rangle_x|1\rangle_z)$, where $|k\rangle_x$ [$|k_z\rangle$] denote eigenstates of σ_x [σ_z], respectively. (ii) The action of a depolarizing channel $\mathcal{E}_k$ on a graph state can equivalently be described by a map $\mathcal{M}_k$ whose Kraus operators contain only products of Pauli matrices σ_z and the identity, where here σ_z may act on particles k and its neighbors, i.e., particles which are (in the corresponding graph) connected by edges to particle k. For a linear cluster state, $\mathcal{M}_k\rho_G = p(t)\rho_G + \{[1 - p(t)]/4\} \times \sum_{j=0}^{3} A_j\rho_G A_j^\dagger$, with $A_0 = \mathbb{1}$, $A_1 = \sigma_z^{(k-1)}\sigma_z^{(k+1)}$, $A_2 = \sigma_z^{(k-1)}\sigma_z^{(k)}\sigma_z^{(k+1)}$, $A_3 = \sigma_z^{(k)}$. This follows from $\sigma_x^{(j)}|\Psi_{\vec{\mu}}\rangle = (-1)^{\mu_j}\sigma_x^{(j)}K_j|\Psi_{\vec{\mu}}\rangle$, where $\sigma_x^{(j)}K_j = \sigma_z^{(j-1)}\sigma_z^{(j+1)}$, and similarly for $\sigma_y^{(j)}$.

A sufficient condition when bipartite entanglement between two neighboring particles, say, particle k and $k+1$, can be created from $\rho(t)$ can be found straightforwardly. To this aim, one performs measurements in the eigenbasis of σ_z on all but particles k and $k+1$. From properties (i) and (ii) follows that these measurements commute with the action of the map on the cluster state. The resulting state can thus be described by $\mathcal{M}_1\mathcal{M}_2\cdots\mathcal{M}_N|\tilde{\Phi}\rangle_{k,k+1}\langle\tilde{\Phi}| \otimes |\chi\rangle\langle\chi|$, where $|\chi\rangle$ is a state of the remaining $(N-2)$ particles, and $|\tilde{\Phi}\rangle$ is a maximally entangled state equivalent up to σ_z operations to $|\Phi\rangle$. It is important to note that $\mathcal{M}_j$ acts trivially on particles k and $k+1$ if $j \notin \{k-1, k, k+1, k+2\}$ since any kind of errors in graph states effect only the corresponding particles and/or its neighbors [see (ii)]. This implies that the resulting state $\rho_{k,k+1}$ of particles k and $k+1$ after tracing out the remaining particles can be obtained by considering only the (reduced) action of $\mathcal{M}_{k-1}, \mathcal{M}_k, \mathcal{M}_{k+1}, \mathcal{M}_{k+2}$ on the state $|\tilde{\Phi}\rangle\langle\tilde{\Phi}|$. One has that $\rho_{k,k+1}$ is distillable if its partial transpose is nonpositive [11] and finds a threshold value $p_< = 0.7167$ ($\kappa t_< = 0.3331$). We emphasize that this threshold value is independent of N, as only errors acting on the direct neighborhood of the pair of particles in question influences the threshold value. Since this method allows one to distill maximally entangled pairs between arbitrary pairs of neighboring particles, we have that for $p \geq p_<$ the linear cluster state remains N-party distillable, independent of the number of particles N.

We point out that one can apply the same reasoning as used above to show that similar results hold for *all* graph states associated with some lattice geometry [7]. If we consider, for instance, a family of graph states whose maximum degree does not increase with the number of particles N and two neighboring vertices k [j] of the corresponding graph with disjoint sets of $n_k + 1$ [$n_j + 1$] neighbors, respectively, one can distill a maximally entangled pair between the two parties if $p^2/4[(1 + p^{n_k}) \times (1 + p^{n_j})] + (1 - p^2)/4 > 1/2$. It turns out that for $\kappa t \leq \ln(\frac{1}{2})/[(\frac{d_k + d_j - 2}{2}) + 2]$, where d_i is the degree of vertex i, the state is certainly distillable, independent of N. For cluster states corresponding to a regular 2D (3D) lattice one finds $p_< = 0.8281$ ($\kappa t_< = 0.1886$) [$p_< = 0.8765$ ($\kappa t_< = 0.1318$)], respectively. This dependence on the degree is consistent with the results obtained for GHZ states, which correspond to a graph with edges $(1, k)$ $\forall$ k. The degree of this graph increases with N and thus the lifetime decreases.

Further generalizations of these results are possible (see [7] for details). One can show that not only graph states but all states produced by an Ising-type interaction, turned on for time $t \neq 0$, whose corresponding (weighted) interaction graphs are associated with some lattice geometry, have a lifetime of distillable entanglement which is independent of the size of the system. This follows from the fact that measurements in the z basis can effectively decouple sets of systems connected in the interaction graph. Furthermore, the scaling behavior of the lifetime of entanglement with the size of the system turns out to be largely independent of the specific decoherence model used. In particular, a similar behavior is found for all individual couplings of particles to an environment described by a quantum optical master equation of the general Lindbladt form. In addition, for all those decoherence models which correspond to noise which acts locally (in the sense that all Kraus operators A_i act nontrivially on a finite, localized number of subsystems), one again finds that the lifetime of cluster and similar states is independent of the size of the system N.

In the following we investigate the effective size of the system that remains entangled during the decoherence process. The fact that N-party entanglement vanishes after a certain time does not imply that all entanglement has disappeared. We consider partitions of the N-particle state into M groups, where now parties within one group are allowed to perform joint operations. For states associated with some lattice geometry, specific partitions correspond to a rescaling of the size of the subsystems. An N-particle state is called M-party distillable entangled if there exists some partitioning into M parties (M partitioning) such that the state is distillable to some true M-party entangled pure state [4]. We will be interested in the lifetime of M-party entanglement.

For GHZ states, one can determine analytically the lifetime of M-party entanglement for any M. Recall the condition for positivity of partial transposition with respect to a group of k parties, $p^N \leq 2\lambda_k$. Since $\lambda_j \geq \lambda_i$ for $j \leq i$, we have that the group containing the fewest number of parties determines the threshold value for which the state is no longer M-party distillable, as the corresponding partial transposition is the first one to become positive. Thus M-party entanglement corresponding to a specific M partitioning has longest lifetime if all groups have (approximately) the same size. For a minimal group size of m particles, an N-particle GHZ state can contain at most $M \equiv [N/m]$ such groups of size m. This allows one to obtain the maximum lifetime of M-party entanglement which is determined by $p^N \leq 2\lambda_m$.

One obtains an upper bound on the lifetime of M-party entanglement if one approximates λ_m by some $\tilde{\lambda}_m < \lambda_m$ and chooses $\tilde{\lambda}_m \equiv (1-p)^m(1+p)^{N-m}/2^{N+1}$. In this case, the partial transposition with respect to a group of m parties is certainly positive if $2\tilde{\lambda}_m \geq p^N$, which can be rewritten as $m \leq N \log[2p/(1+p)]/\log[(1-p)/(1+p)]$. Using that $M = [N/m]$, we have that an N-particle state is no longer M-party entangled if

$$M \geq [\log(1-p) - \log(1+p)]/[\log(2p) - \log(1+p)]. \tag{2}$$

On the one hand, Eq. (2) [illustrated in Figs. 1(b) and 1(c)] provides an upper bound on the lifetime $\kappa\tau_M$ of M-party entanglement in the system. On the other hand, for a fixed time t, Eq. (2) allows one to determine the maximum M of distillable multiparty entanglement remaining in the system. One observes [see Figs. 1(b) and 1(c)] that the maximum M rapidly decreases with t. For $\kappa t \ll 1$, one finds that $M \approx -2\log(\kappa t)/(\kappa t)$, while for $\kappa t > 0.8049$ we have that also 2-party entanglement disappears and the state becomes fully separable as all partial transposes are positive (which is a sufficient condition for separability for such states [4]). We also point out that this bound on the maximal size of M-party entanglement is independent on the number of particles N. That is, even if one considers groups of size $m \to \infty$ (for a total number of particles $N \to \infty$), the maximum number M of such groups which remain entangled after a certain time t is finite. While for the finest partition of the system into N parties the lifetime $\kappa\tau_N$ essentially scales like $1/N$, for the coarsest partition of the N parties into two groups we have no scaling behavior with N, i.e., $\kappa\tau_2 =$ const. It follows that in the limit $N \to \infty$ any partitioning in groups of finite size m leads to a vanishing lifetime of the corresponding $M = N/m$ party entanglement. Only if one considers a fixed number of groups M whose size $m = N/M$ grows to infinity as $N \to \infty$, i.e., the groups are of macroscopic size, one obtains a finite lifetime of the corresponding M-party entanglement.

We remark that Eq. (2) also allows one to determine the lifetime of genuine N-party entanglement, obtained from $M = N/m$ by setting $m = 1$. In a similar way, one can obtain a lower bound on the lifetime of M-party entanglement by choosing $\tilde{\lambda}'_m \equiv 2(1-p)^m(1+p)^{N-m}/2^{N+1} > \lambda_m$. One finds that for $M \leq \log[2(1-p)/(1+p)]/\log[2p/(1+p)]$ all partial transposition with respect to this M partitioning are certainly nonpositive, which for these kinds of states already ensures that the state is M-party distillable [4]. The condition for 2-party entanglement has recently been derived by Simon and Kempe in Ref. [8], who observed that the threshold value for p with respect to the partition $N/2 - N/2$ decreases with the size of the system. Based on this observation, they conclude that GHZ states of more particles are more stable against local decoherence. However, as pointed out in this Letter, the effective number of subsystems that remain entangled decreases with time, such that the entanglement becomes bipartite when approaching the threshold value found by Simon and Kempe. The lifetime of genuine N-party entanglement thus decreases in fact with the size of the system.

On the other hand, for cluster states (and similar graph states) one can show that there is no scaling with respect to either the size of the partitions or N. To be specific, for any cluster state there exist times $t_>$, $t_<$ independent of N such that for $t \geq t_>$ the state $\rho(t)$ is separable with respect to the finest partition (and hence not distillable with respect to any partition), while for $t < t_<$ the state is distillable with respect to the finest partition (and hence distillable with respect to any partition). While the existence of such a time $t_<$ was already shown earlier in this Letter, one can prove [7] that $\rho(t)$ is fully separable for $\kappa t \geq \kappa t_>$ with $\kappa t_> = -2m\ln(\sqrt{2}-1)$, where m is the degree of the graph.

In this Letter we studied the behavior of multiparticle entangled states under decoherence. For GHZ states, we found that the lifetime of true M-party entanglement decreases with the size of the system and the effective number of entangled subsystems decreases with time. For cluster and similar graph states, however, we have shown that the lifetime of M-party entanglement is independent of the size of the system. These results suggests that true multiparticle entanglement in macroscopic objects can be more stable (and might be more common) than previously thought.

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