

Theory for Explosive Ideal Magnetohydrodynamic Instabilities in Plasmas

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Flux tubes confined in tokamaks are observed to erupt explosively in some plasma disruptions and edge localized modes. Similar eruptions occur in astrophysical plasmas, for example, in solar flares and magnetospheric substorms. A single unifying nonlinear evolution equation describing such behavior in both astrophysical and tokamak plasmas is derived. This theory predicts that flux tubes rise explosively, narrow, and twist to pass through overlying magnetic field lines without reconnection.

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There are numerous examples of explosive instabilities in plasmas, both astrophysical (e.g., solar flares and prominence eruptions [1,2] and magnetospheric substorms [3]) and laboratory (e.g., disruptions [4] and edge localized modes, or ELMs, [5,6] in tokamaks). There is strong evidence that at least some of these events are triggered by eruptions of flux tubes through the plasma (see, for example, [7–10]). Because these systems are evolving slowly through the stability threshold (prior to the event), it is difficult to explain fast growth without nonlinearity. In this Letter we develop a nonlinear ideal magnetohydrodynamic (MHD) theory of explosive flux tube eruptions in a tokamak that is close to the linear stability threshold. We extend the original work of Hurricane *et al.* [11], which was applied to solar flares [12] and magnetospheric substorms [13], to include key aspects of toroidal geometry. The astrophysical and laboratory phenomena are then found to be governed by a single, unifying one parameter nonlinear equation. Solutions are qualitatively independent of this parameter—specifically all flux tubes rise explosively, narrow, and twist as they pass through the overlying plasma. We therefore predict flux tube eruptions in all situations to be similar and may be driven by the same basic mechanism. As well as being of basic scientific interest, this is of extreme practical importance, as large ELMs and disruptions may, if not avoided, cause structural damage in planned burning plasma tokamaks.

The instability on which we focus is called the MHD ballooning mode (see [14], for example), which is driven unstable in a plasma when the pressure gradient exceeds a critical value. Let us review the features of the linear theory, which is helpful for understanding the nonlinear theory. The most unstable modes have a high toroidal mode number, n , and therefore a short wavelength perpendicular to the field line in the flux surface, $\sim a/n$ where a is the minor radius of the tokamak plasma. On the other hand, to minimize field line bending, the wavelength of the perturbation along the field line is very long,

comparable with the equilibrium plasma scale lengths. To describe such structures and yet satisfy the necessary periodicity constraints, the equations are solved on an extended poloidal angle (θ) domain $[-\infty, \infty]$ from which periodic solutions are constructed [14]. To leading order in $1/n^{1/2}$ one obtains the ballooning equation that determines the mode structure along, but not across, the field line. The growth rate is an eigenvalue of this equation. At higher order, $O(n^{-1})$, one finds that the radial mode structure spans a distance $\sim a/n^{1/2}$ [14].

Let us consider some of the properties of the “ballooning” equation close to marginal stability, i.e., when the growth rate γ (normalized to the Alfvén frequency) is small. Inertia can be neglected at distances along the field line, such that $|\gamma\theta| \ll 1$ (we call this the “ideal region”). For $1 \ll |\theta| \ll 1/\gamma$, the “radial” displacement, ξ_ψ , has the form [14]

$$\lim_{1 \ll |\theta| \ll 1/\gamma} \xi_\psi = A_\pm \left(\frac{1}{|\theta|^{\lambda_L}} + \frac{\Delta'_\pm}{|\theta|^{\lambda_S}} \right). \quad (1)$$

A_\pm and Δ'_\pm are constants and \pm indicates the sign of θ . The fractional indices are the Mercier indices, $\lambda_{L,S} = 1/2 \mp \sqrt{1/4 - D_M}$, where D_M is the well-known Mercier coefficient and is a property of the equilibrium, particularly the shape. For typical tokamak situations, away from the core, $D_M < 0$, so that $\lambda \equiv \lambda_S - \lambda_L > 1$.

Inertia cannot be neglected in the regions $\gamma\theta \gtrsim 1$ and $\gamma\theta \lesssim -1$. The form of the ballooning equation in these “inertial regions” indicates that the solution must be a function only of the variable $v \equiv \gamma\theta$ —i.e., $\xi_\psi = \xi_\psi(v)$. However, for $|\gamma\theta| \ll 1$ it must match the ideal region solution from Eq. (1). Thus, it must have the form

$$\lim_{\gamma|\theta| \rightarrow 0} \xi_\psi = \hat{A}_\pm \left(\frac{1}{(\gamma|\theta|^{\lambda_L})} + \frac{b_\pm}{(\gamma|\theta|^{\lambda_S})} \right). \quad (2)$$

Solving the inertial region equations with the boundary condition that $\xi_\psi \rightarrow 0$ at $\theta = \pm\infty$, the numbers b_\pm are obtained [15]. Because the inertial region equations are

symmetric, $b_+ = b_- = b$. Matching the solutions in Eqs. (1) and (2) yields $\Delta'_+ = \Delta'_- = \Delta'$, which can then be determined entirely from the ideal region. In addition, the matching determines the growth rate: $(\Delta')^{-1} \equiv \delta = (\gamma^\lambda)/b$. There is a subtlety here. Close to marginal stability δ is small and, if we were to retain inertia in the ideal region, this would lead to a correction, which is $\mathcal{O}(\gamma^2)$. Then we would obtain

$$\delta_0 + \delta_1 \gamma^2 = \frac{\gamma^\lambda}{b}. \quad (3)$$

Equation (3) is a key result in defining our formalism. Consider first the tokamak situation in which the shaping of the cross section is not too strong, so that $\lambda < 2$. Then the term in $\delta_1 \gamma^2$ can be neglected, and inertia can be ignored when calculating Δ' from the ideal region. Matching this Δ' to (causal) solutions in the inertial region determines the growth. In the opposite case, when $\lambda > 2$, the term in $\delta_1 \gamma^2$ is the dominant inertial correction, and the effect of the inertial region can be neglected. In this situation one could apply line-tied boundary conditions at $|\theta| \gg 1$ and retain inertia in the ideal region. This is then equivalent to the ballooning mode analysis in [11].

Let us now turn to the nonlinear theory in the situation when $\lambda < 2$, which is the main purpose of this Letter. We shall find that our nonlinear mode is localized about a particular field line so that periodicity is not important provided that the perturbed region does not close on itself, but wraps many times around the torus. Thus, we ignore the periodicity constraint and the extended angle, θ , is then reinterpreted as a measure of the distance along an infinitely long field line. The calculation in the ideal region then broadly follows that of [11], but differs in a number of crucial ways. To make analytic progress, it is necessary to identify a small parameter so that we can develop an expansion procedure to solve the full, nonlinear ideal MHD equations. The small parameter we employ, ϵ , is a measure of the localization of the mode perpendicular to the magnetic field line. We use the coordinates, θ , ψ , the poloidal magnetic flux, and α , an angular coordinate perpendicular to the equilibrium magnetic field such that $\mathbf{B}_0 = \nabla\psi \times \nabla\alpha$. Derivatives of perturbed quantities with respect to these coordinates are ordered:

$$\left. \frac{\partial}{\partial \theta} \right|_{\alpha, \psi} \sim \epsilon^0, \quad \left. \frac{\partial}{\partial \psi} \right|_{\alpha, \theta} \sim \epsilon^{-1}, \quad \left. \frac{\partial}{\partial \alpha} \right|_{\theta, \psi} \sim \epsilon^{-2}. \quad (4)$$

Note that this ordering reflects the linear mode structure, in which case we could identify $\epsilon \sim n^{-1/2}$.

The Lagrangian plasma displacement is

$$\xi = \xi_\psi \mathbf{e}_\perp + \xi_\alpha \mathbf{e}_\parallel + \xi_\parallel \mathbf{B}_0, \quad (5)$$

where $\mathbf{e}_\perp = \nabla\alpha \times \mathbf{b}_0$ and $\mathbf{e}_\parallel = \mathbf{b}_0 \times \nabla\psi$ with $\mathbf{b}_0 = \mathbf{B}_0/B_0$. Anticipating cancellation in $\nabla \cdot \xi$, we order $\xi_\parallel \sim \xi_\psi$ and $\xi_\alpha \sim \epsilon \xi_\psi$. An absolute ordering of $\xi_\psi \sim \epsilon^2$ and $\partial/\partial t \sim \epsilon^{2/\lambda}$ is chosen so as to introduce nonlinearity and time evolution in the equation determining the perpendicular mode structure. With this ordering $\xi \cdot \nabla \ll 1$, and so we avoid shocks. We apply this ordering to the Lagrangian MHD force balance equation [11].

Our first task is to calculate the nonlinear Δ' from the ideal region. We find that the perturbation is incompressible to three orders of ϵ . From the second order force equation in the \mathbf{e}_\perp direction we obtain

$$\xi_\psi^{(2)} = \hat{\xi}(\psi, \alpha, t)H(\theta), \quad (6)$$

where a superfix on perturbed quantities indicates the order in ϵ , e.g., $\xi_\psi = \sum_j \epsilon^j \xi_\psi^{(j)}$. The variation along the field H is determined from the ballooning equation:

$$B_0 \mathbf{B}_0 \cdot \nabla_0 \left[\frac{|\mathbf{e}_\perp|^2}{B_0^2} [\mathbf{B}_0 \cdot \nabla_0 (B_0 H)] \right] + \frac{2\mu}{B_0^2} (\mathbf{e}_\perp \cdot \kappa_0) (\mathbf{e}_\perp \cdot \nabla_0 p_0) H = 0, \quad (7)$$

where $\kappa_0 = \mathbf{B}_0 \cdot \nabla \mathbf{B}_0$ and p_0 is the pressure. We have inserted a fictitious eigenvalue, μ , to provide a well defined system to solve for H with boundary conditions $H \rightarrow 0$ as $\theta \rightarrow \pm\infty$. If we were precisely at marginal stability, then $\mu = 1$. In general, we are interested in situations that are only just unstable, in which case μ is slightly less than 1. To compensate, a correction $\sim(1 - \mu)$ is reintroduced in the higher order equations. A final point to note is that the equilibrium is invariant under the transformation $\alpha \rightarrow \alpha + f(\psi)$, and, making this transformation, we find $\mu = \mu(df/d\psi)$.

We now proceed to the higher order equations to determine an equation for the envelope function $\hat{\xi}$ and $f' \equiv df/d\psi$. At third order we determine the small compressibility and are forced to choose f' to minimize μ . We obtain our nonlinear equation from the fourth order curl of the perpendicular force balance, which involves $\xi^{(4)}$. One solubility condition for $\xi^{(4)}$ is obtained by multiplying this equation by H and integrating out to large distance $1 \ll |\theta| \ll \gamma^{-1}$ along the field line. After appropriate integrations by parts and substantial algebra the $\xi^{(4)}$ terms are reduced to a boundary value term that is proportional to $(\Delta')^{-1}$. The result is our final equation for Δ' in the ideal region. It takes the relatively simple form for up-down symmetric equilibria:

$$-\frac{(A_+^2 + A_-^2)}{\Delta'(\alpha, \psi, t)} \lambda \frac{\partial^2 u}{\partial \alpha^2} = C_1 \left[2(1 - \mu) \frac{\partial^2 u}{\partial \alpha^2} - \frac{\partial^2 \mu}{\partial f'^2} \frac{\partial^2 u}{\partial \psi^2} \right] + C_2 \frac{\partial}{\partial \alpha} \left[\left(\frac{\partial u}{\partial \alpha} \right)^2 \right] + C_4 \frac{\partial^2 u}{\partial \alpha^2} \frac{\partial^2}{\partial \psi^2} \left(\frac{\partial u}{\partial \alpha} \right)^2. \quad (8)$$

We have replaced $\hat{\xi} = \partial u / \partial \alpha$, and the overbar denotes an integral of the quantity over the α variable from $-\infty$ to $+\infty$. The coefficients, C_j , represent averages along the field line (to infinity) involving the function H , which are similar to those given in Hurricane *et al.*; details will be given in a future paper.

We leave interpretation of the various terms in this equation until after we have addressed the solution in the inertial region. The radial component of the displacement at the low $\gamma\theta$ end of this region scales as $\xi_{\psi} e_{\perp} \sim \theta^{1-\lambda_S}$. Thus, provided $\lambda_S > 1$ (i.e., $D_M < 0$), the displacement and the nonlinearities are small in the inertial region. For $\lambda_S \leq 1$ nonlinearities are important along the full length of the field line, including the inertial region. This reveals itself in our analysis in the divergence of the coefficient C_2 for $\lambda_S \leq 1$. Typically this happens only near the center of a tokamak. Thus, for tokamak disruptions or ELMs, we restrict consideration to $\lambda_S > 1$.

The equations describing the evolution in the inertial region are therefore a pair of linear, coupled differential

equations for the radial perturbation and $\nabla \cdot \xi$ (plasma compressibility must be retained). These equations are solved using a Laplace transform in time. This is similar to the linear procedure, with the Laplace transform variable p replacing the linear growth rate, γ . The expression (2) with $\gamma \rightarrow p$ and $\hat{A} = \hat{A}(p)$ (and b known) holds in the matching region. Inverting the Laplace transform and matching the powers of θ in the asymptotic form, Eq. (1), of the ideal region yields the expression

$$\frac{1}{\Delta'} \frac{\partial \hat{\xi}(\alpha, \psi, t)}{\partial \alpha} = \frac{1}{b\Gamma} \frac{\partial^2}{\partial t^2} \frac{\partial}{\partial \alpha} \left[\int_0^t dt' \frac{\hat{\xi}(\alpha, \psi, t')}{(t-t')^{\lambda-1}} \right], \quad (9)$$

where $\Gamma = \Gamma(2 - \lambda)$. Equation (9) is valid over the full regime of relevance, $1 < \lambda < 2$.

All that remains is to match the expressions for Δ' , which we have obtained in the two regions. Thus, combining Eqs. (8) and (9), and normalizing variables appropriately, we derive a nonlinear evolution equation that depends only on D_M :

$$\frac{\partial^2}{\partial t^2} \frac{\partial^2}{\partial \alpha^2} \left[\int_0^t dt' \frac{u(\alpha, \psi, t')}{(t-t')^{\lambda-1}} \right] = \left[2(1 - \mu) \frac{\partial^2 u}{\partial \alpha^2} - \frac{\partial \mu}{\partial f'^2} \frac{\partial^2 u}{\partial \psi^2} \right] + \frac{\partial}{\partial \alpha} \left[\left(\frac{\partial u}{\partial \alpha} \right)^2 \right] + \frac{\partial^2 u}{\partial \alpha^2} \frac{\partial^2}{\partial \psi^2} \left(\frac{\partial u}{\partial \alpha} \right)^2. \quad (10)$$

Equation (10) describes the early nonlinear evolution of tokamak plasmas in most situations (i.e., $1 < \lambda < 2$). The term on the left-hand side is the formal representation of a fractional derivative and in the limit $\lambda \rightarrow 2$ it matches smoothly onto the second derivative (in t). For $\lambda > 2$ we must use the equation derived in Hurricane *et al.* [11] where the time integral in the brackets on the left-hand side of Eq. (10) is replaced with $u(\alpha, \psi, t)$.

Let us now consider the various drive terms. The first two terms on the right-hand side are standard linear terms. The first represents the linear drive for the unstable flux tube and close to linear marginal stability is positive (unstable) in a narrow region of ψ . The second represents the stabilizing terms that come from pushing aside flux tubes to allow the exploding tube through (it is smaller the narrower the tube in α). For small amplitude the displacement is confined to the narrow linearly unstable region [14]. The term that is quadratic in u is the main nonlinear drive for the instability, and represents a weakening of the field line bending of expanding flux tubes. Balancing this against the inertia provides an estimate for $\hat{\xi} (\equiv \partial u / \partial \alpha)$ in the nonlinear evolution phase:

$$\hat{\xi}(\alpha, \psi, t) \sim \frac{1}{[t_0(\alpha, \psi) - t]^{\lambda}}. \quad (11)$$

The time t_0 depends on the initial conditions. It is interesting to note that this exhibits explosive growth as $t \rightarrow t_0$, and matches smoothly onto the line-tied result [16] as $\lambda \rightarrow 2$. Since the peak of the displacement (at the minimum of t_0) in Eq. (11) grows fastest, this nonlinearity tends to narrow the mode. The cubic nonlinearity repre-

sents the flattening of the pressure in the vicinity of the mode. This term does not significantly affect the explosive nature of the instability, but does influence the radial mode structure. Note that it is negative (stabilizing) around the region of maximum linear instability, but positive (destabilizing) at larger radial distances from the mode. Thus, this term simply acts to make the mode more radially extended than linear theory would predict. Balancing the two nonlinear terms, we find that $(\Delta\psi)^2 / (\Delta\alpha) \sim \xi \sim (t_0 - t)^{-\lambda}$ where $\Delta\alpha$ and $\Delta\psi$ are the flux tube width and the radial extent, respectively. Thus, the competition between the quadratic and cubic nonlinearities leads to a narrowing in α and a broadening (beyond the linearly unstable region) in ψ .

We solve Eq. (10) numerically for $\lambda = 1.6$. In the linear regime the displacement grows exponentially in the linearly unstable region. At larger amplitude the displacement grows explosively as $(t_0 - t)^{-1.7}$ with $t_0 = 10.5$, i.e., broadly consistent with that given by Eq. (11). In Fig. 1(a) we show the evolution of the instantaneous growth rate and the width $\Delta\alpha$. In Fig. 1(b) we show a picture of the perturbed flux surfaces at time ~ 10 —the exploding flux tube exhibits itself as a narrowing finger. Of course, the orderings assumed in deriving Eq. (10) break down close to the singularity. Nevertheless, the calculation does demonstrate that the ballooning mode grows very much faster than linear theory would predict close to marginal stability.

In conclusion, we have extended the theory of nonlinear ballooning instabilities to the case of plasma systems with toroidal symmetry, such as tokamaks. In doing

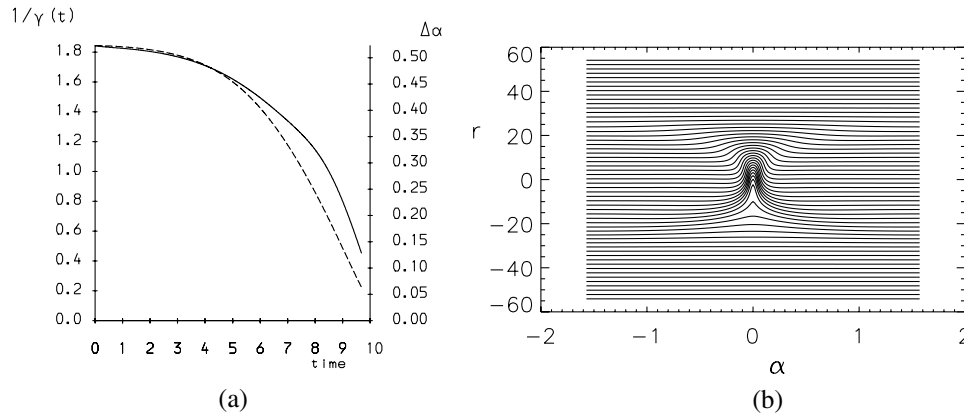


FIG. 1. (a) Plots of the inverse of the instantaneous growth rate, $\gamma(t)$ (full curve), and the width $\Delta\alpha$ of the perturbation (dashed curve) as a function of time. (b) Plot of constant flux surface contours in the plane of minor radius, r , and α .

so, we have identified three different regimes: (1) $D_M < -3/4$, the displacement is not extended far along the field line, and the theory developed by Hurricane *et al.* [11] is appropriate; (2) $0 > D_M > -3/4$, nonlinearities are important at small distances along field lines while inertia is important only at large distances and the theory developed here is appropriate; (3) $D_M > 0$, both nonlinearities and inertia are important at large distances along the field lines—this theory is not yet developed. In most tokamak discharges D_M is positive in a small central region and the condition $0 > D_M > -3/4$ holds over most of the flux surfaces: $D_M < -3/4$ is usually associated with strong triangular shaping.

In a tokamak, the theory predicts that the ballooning instability will eject a flux tube of hot plasma into the colder edge plasma region, the so-called scrape-off layer. This flux tube is connected back to the hot plasma. Heat flow along this tube could provide a mechanism for the rapid heat loss observed in both the ELM and the disruption. The actual mechanism by which the heat and particles leave the hot plasma flux tube and enter the scrape-off layer is not determined by the ideal MHD theory presented here, and is the subject of future research. It is interesting to note that, like inertia, nonideal MHD physics is usually important only far along the field line. Therefore our calculation of the effects of the nonlinearities on Δ' is likely to be useful for a wide range of nonideal MHD models. We have presented a theory that provides a unified description of flux tube eruptions in laboratory and astrophysical plasmas. Much work is needed to tie the theory to actual observations.

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