Simplified Test of Universality in Lattice QCD

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A simplified test of universality in lattice QCD is performed by analytically evaluating the continuous Euclidean time limits of various lattice fermion determinants, both with and without a Wilson term to lift the fermion doubling on the Euclidean time axis, and comparing them with each other and with the zeta-regularized fermion determinant in the continuous time–lattice space setting. The determinant relations expected from universality considerations are found to be violated by a certain gauge field-dependent factor; i.e., we uncover a "universality anomaly." The physical significance, or lack thereof, of this factor is a delicate question that remains to be settled.

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The low-lying spectrum of the naive lattice Dirac operator approximates the low-lying spectrum of the continuum Dirac operator but with a 16-fold degeneracy due to fermion doubling [1]. Lattice QCD (LQCD) with a naive fermion is therefore regarded as a regularization of continuum QCD with 16 degenerate fermion flavors. Lattice QCD with a staggered fermion [2] is regarded as a regularization of four flavor QCD, in accordance with the fact that one naive fermion flavor is equivalent after spin diagonalization to four staggered fermion flavors [3]. Lattice QCD with a Wilson fermion [1], where the Wilson term is added to the naive fermion action to lift the fermion doubling, is regarded as a regularization of QCD with a single fermion flavor. Implicit in this is a *universality hypothesis:* the LQCD's with naive fermion, staggered fermion, and Wilson fermion are all in the right universality class to reproduce continuum QCD, the only difference being in the number of continuum fermion flavors the different lattice fermion formulations describe.

It is highly desirable to test this universality hypothesis wherever possible. This is particularly important in view of the fact that LQCD calculations with both dynamical Wilson and staggered fermions are currently being pursued at great effort and expense [4]. An interesting quantity to consider in this context is the fermion determinant, which appears in the QCD functional integral when the fermions are dynamical. In LQCD with dynamical staggered fermions, the fourth root of the staggered fermion determinant is used as the fermion determinant for a single quark flavor. An important test of the universality hypothesis is therefore to check whether the fourth power of the Wilson fermion determinant coincides with the staggered fermion determinant in the continuum limit or, equivalently, whether the 16th power of the Wilson determinant coincides with the naive fermion determinant in this limit. Such a test appears analytically impossible with currently known techniques though. However, a simplified version of this test is feasible: instead of the full continuum limit one can take the continuum limit for one of the spacetime coordinates while keeping the remaining coordinates discrete. In this Letter we evaluate the continuous Euclidean time limits of the various lattice fermion determinants, both with and without the time part of the Wilson term in the action, and compare them with each other and with the zeta-regularized fermion determinant in the continuous time–lattice space setting.

On a finite spacetime lattice, with N_β sites along the Euclidean time axis, a the time lattice spacing, a' the spatial lattice spacing, $\beta = aN_\beta$ the time length (inverse temperature in the finite temperature QCD setting), we consider lattice fermion actions of the form $S = a(a')^3\sum_{(\mathbf{x},\tau)} \overline{\psi}(\mathbf{x},\tau)D^{(r)}\psi(\mathbf{x},\tau)$, where $(\mathbf{x},\tau)$ runs over the lattice sites and

$$D^{(r)} = \gamma_4\frac{1}{a}\nabla_4 + \frac{r}{2a}\Delta_4 + D_{\mathrm{space}} + m, \tag{1}$$

$$\nabla_4\psi(\mathbf{x},\tau) = \frac{1}{2}[U_4(\mathbf{x},\tau)\psi(\mathbf{x},\tau+a) - U_4(\mathbf{x},\tau-a)^{-1}\psi(\mathbf{x},\tau-a)], \tag{2}$$

$$\Delta_4\psi(\mathbf{x},\tau) = 2\psi(\mathbf{x},\tau) - U_4(\mathbf{x},\tau)\psi(\mathbf{x},\tau+a) - U_4(\mathbf{x},\tau-a)^{-1}\psi(\mathbf{x},\tau-a), \tag{3}$$

$$D_{\mathrm{space}}^{(r')} = \frac{1}{a'}\nabla\!\!\!\!/_{\mathrm{space}} + \frac{r'}{2a'}\Delta_{\mathrm{space}}, \tag{4}$$

$$\nabla\!\!\!\!/_{\mathrm{space}}\psi(\mathbf{x},\tau) = \sum_{\sigma=1}^{3}\gamma_\sigma\frac{1}{2}[U_\sigma(\mathbf{x},\tau)\psi(\mathbf{x}+\hat{\sigma},\tau) - U_\sigma(\mathbf{x}-\hat{\sigma},\tau)^{-1}\psi(\mathbf{x}-\hat{\sigma},\tau)], \tag{5}$$

$$\Delta_{\mathrm{space}}\psi(\mathbf{x},\tau) = \sum_{\sigma=1}^{3}[2\psi(\mathbf{x},\tau) - U_\sigma(\mathbf{x},\tau)\psi(\mathbf{x}+\hat{\sigma},\tau) - U_\sigma(\mathbf{x}-\hat{\sigma},\tau)^{-1}\psi(\mathbf{x}-\hat{\sigma},\tau)]. \tag{6}$$

For $r = r' = 0$ this is the naive fermion action, while for $r = r' \neq 0$ it is the Wilson action. We shall evaluate the continuous time limit ($a \to 0$, $N_\beta \to \infty$ with $\beta = aN_\beta$ held fixed) of the fermion determinants $\det D_a^{(r)}$ at $r = 0$ and $r = 1$ and compare them with each other and with the zeta-regularized fermion determinant $\det_\zeta D_\alpha$ in the continuous time–lattice space setting. The Dirac operator in

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the latter setting is

$$D = \gamma_4\left(\frac{d}{d\tau} + A_4\right) + D_{\rm space} + m, \qquad (7)$$

with $A_4(\mathbf{x}, \tau)$ being the 4-component of a smooth continuum gauge field such that $U_4(\mathbf{x}, \tau) = Te^{\int_0^1 aA_4[\mathbf{x}, \tau+(1-t)a]\,dt}$ ($T = t$ ordering) is the lattice transcript. The subscripts "α" in $D_\alpha^{(r)}$ and D_α refer to the operators defined by replacing $U_4 \to e^{-\alpha a}U_4$ in (1)–(6) and $A_4 \to A_4 - \alpha$ in (7), respectively. The role of the complex parameter α is to incorporate the effect of a general boundary condition (BC) at the time boundaries: $D_\alpha^{(r)}$ (D_α) with *periodic* time BC has the same spectrum and determinant as $D^{(r)}$ (D) with time BC $\psi(\mathbf{x}, \beta) = e^{\alpha\beta}\psi(\mathbf{x}, 0)$. Thus, the introduction of α allows us to always take periodic time BC when considering the fermion determinant. It can also be used to incorporate a chemical potential μ: QCD at finite temperature and density, where the fermion fields satisfy antiperiodic time BC, corresponds to $\alpha = \mu + i\pi/\beta$. The gauge fields are required to satisfy periodic time BC. The spatial BC's for the fermion and gauge fields do not play a role in our considerations and are left unspecified.

The term $\frac{r}{2a}\Delta_4$ in (1) is the "time part" of the usual Wilson term. It lifts the fermion doubling on the Euclidean time axis when $r \neq 0$. Therefore, if we think of the continuous time–lattice space setting as the "continuum setting," then the aforementioned universality hypothesis, which relates the continuum limits of the naive, staggered, and Wilson fermion determinants, translates into the *simplified universality hypothesis:*

$$\lim_{a\to 0} \det D_\alpha^{(0)} = (\lim_{a\to 0} \det D_\alpha^{(r)})^2_{r\neq 0} \qquad (\mathrm{mod\,PIFs}), \qquad (8)$$

where PIFs refers to physically inconsequential factors. This is now something that can be checked analytically. Our main technical result in this Letter is

$$\lim_{a\to 0} \det D_\alpha^{(0)} = (\lim_{a\to 0} \det D_\alpha^{(1)})^2 e^{-\int_0^\beta \mathrm{Tr}[\frac{r'}{2a'}\Delta_{\rm space}(\tau)]\,d\tau} \qquad (\mathrm{mod\,PIFs}), \qquad (9)$$

where $\Delta_{\rm space}(\tau)$ is defined on the space of lattice spinor fields $\psi(\mathbf{x})$, living only on the spatial lattice, by replacing $\psi(\mathbf{x}, \tau)$ by $\psi(\mathbf{x})$ in (6). The PIFs in (9) are gauge field-independent factors whose only effects are to produce constant (vacuum) shifts in certain physical quantities. They include inverse powers of a, which diverge in the $a \to 0$ limit.

The result (9) reveals a "universality anomaly": the exponential factor in the right-hand side violates the universality expectation (8). Thus, it is important to ascertain the significance, or lack thereof, of this factor. Since it is gauge field dependent, it cannot, strictly speaking, be regarded as a PIF in the continuous time–lattice space theory. However, since the spatial Wilson term $\frac{r'}{2a'}\Delta_{\rm space}$ formally vanishes in the spatial continuum limit, one could argue that the exponential factor is effectively a PIF when one goes on to take that limit. This is a delicate issue though, since $\mathrm{Tr}\frac{1}{2a'}\Delta_{\rm space}$ actually diverges in the $a' \to 0$ limit (the largest eigenvalue of $\frac{1}{2a'}\Delta_{\rm space}$ is $\sim\frac{1}{a'}$). Further study is required to clarify this issue.

Since the reasoning that leads to the universality expectation (8) is the same as that which leads to the expectation that LQCD with naive, staggered, and Wilson fermions are all in the same universality class, Eq. (9) is a potential reason for concern about whether the latter universality hypothesis really holds. It would therefore be a significant reassurance if the anomaly factor in (9) can be shown to be physically inconsequential. It should be noted, however, that even if this turns out not to be the case, it would not in itself invalidate the universality hypothesis for LQCD with naive, staggered, and Wilson fermions, since the comparison between naive fermion ($r = r' = 0$) and Wilson fermion ($r = r' \neq 0$) is not covered by (9), and we could still be lucky in this case. But it would certainly raise a serious concern.

Remarkably, the anomaly in (9) is mirrored by an ambiguity in the zeta-regularized fermion determinant in the continuous time–lattice space setting. The latter can be expressed either as $\det_\zeta D_\alpha$ or $\det_\zeta(\gamma_4 D_\alpha)$ (since $\overline{\psi} = \psi^*\gamma_4$). Formally, the determinants of D_α and $\gamma_4 D_\alpha$ coincide, but it turns out that the rigorously defined zeta determinants do not. In fact, we find $\det_\zeta(\gamma_4 D_\alpha) = e^{-\frac{1}{2}\int_0^\beta \mathrm{Tr}[\frac{r'}{2a'}\Delta_{\rm space}(\tau)]\,d\tau}\det_\zeta D_\alpha$ (mod PIFs), and

$$\begin{aligned} \lim_{a\to 0} \det D_\alpha^{(1)} &= \det{}_\zeta D_\alpha \qquad (\mathrm{mod\,PIFs}), \\ \lim_{a\to 0} \det D_\alpha^{(0)} &= \det{}_\zeta(\gamma_4 D_\alpha)^2 \qquad (\mathrm{mod\,PIFs}). \end{aligned} \qquad (10)$$

This shows that the lattice regularizations are consistent with zeta regularizations of the fermion determinant in the continuous time–lattice space setting, and that the requirement that the anomaly factor in (9) be physically inconsequential is also necessary for consistency of continuous time–lattice space QCD when the fermion determinant is defined by zeta regularization.

In the remainder of this Letter we sketch the derivation of (9) and (10) and give other, more explicit, expressions for the $a \to 0$ limits of $\det D_\alpha^{(0)}$ and $\det D_\alpha^{(1)}$. The full details are provided in [5]. It is convenient to regard $\psi(\mathbf{x}, \tau)$ as a function $\Psi(\tau)$ living on the lattice sites of the Euclidean time axis and taking values in the vector space $W = \{\psi(\mathbf{x})\}$, i.e., the space of lattice spinor fields living on the spatial lattice only. Set $N := \dim W$. Define the linear operator $U_4(\tau)$ on W by $(U_4(\tau)\psi)(\mathbf{x}) = U_4(\mathbf{x}, \tau)\psi(\mathbf{x})$. The operator $D_{\rm space}(\tau)$ on W is defined similarly by replacing $\psi(\mathbf{x}, \tau)$ by $\psi(\mathbf{x})$ in (4)–(6). Since $\Psi(\beta) = \Psi(0)$ we can represent Ψ by the vector $\hat{\Psi} = (\hat{\Psi}(0), \ldots, \hat{\Psi}(N_\beta - 1))$ where $\hat{\Psi}(k) = \Psi(ka)$. Then $D^{(r)}$ is represented by

$$\hat{D}^{(r)}\hat{\Psi}(k) = d_{-1}^{(r)}(k)\hat{\Psi}(k-1) + d_0^{(r)}(k)\hat{\Psi}(k) + d_1^{(r)}(k)\hat{\Psi}(k+1), \qquad (11)$$

where the operators $d_j^{(r)}(k):W \to W$ are given by $d_1^{(r)}(k) = \frac{1}{2a}(\gamma_4 - r)\hat{U}_4(k)$, $d_{-1}^{(r)}(k) = -\frac{1}{2a}(\gamma_4 + r)\hat{U}_4(k-1)^{-1}$, $d_0^{(r)}(k) = \frac{r}{a} + \hat{D}_{\rm space}(k) + m$ with $\hat{U}_4(k) := U_4(ka)$ and $\hat{D}_{\rm space}(k) := D_{\rm space}(ka)$. The generalization of $\hat{D}^{(r)}$ to $\hat{D}_\alpha^{(r)}$, given by $U_4 \to e^{-\alpha a}U_4$, is equivalent to $d_{\pm 1}^{(r)} \to e^{\mp\alpha a}d_{\pm 1}^{(r)}$ in (11). After writing $\hat{D}_\alpha^{(r)}$ as an $N_\beta \times N_\beta$ matrix, its determinant can be straightforwardly evaluated via the method of [6]. The cases $r = 1$ and $r \neq 1$ require separate treatments due to the fact that $d_{\pm 1}^{(r)}(k)$ is invertible when $r \neq 1$ but not when $r = 1$. The details of the calculation are provided in [5]; here we simply quote the results, assuming for convenience that N_β is even in the $r \neq 1$ case:

$$\det D_\alpha^{(r\neq 1)} = \left(\frac{(1-r^2)^2}{2a}\right)^{NN_\beta} e^{-\alpha\beta N} \times \det\left[\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - e^{\alpha\beta}\hat{\mathcal{U}}^{(r)}(N_\beta/2)\right], \quad (12)$$

$$\det D_\alpha^{(1)} = \left(\frac{1}{a}\right)^{NN_\beta} e^{-\alpha\beta N/2}\left[\prod_{k=0}^{N_\beta-1}\det[1 + a\hat{M}(k)]\right]^{1/2} \times \det[1 - e^{\alpha\beta}\hat{\mathcal{V}}(N_\beta)], \quad (13)$$

where $\hat{M}(k) := \frac{r'}{2a'}\hat{\Delta}(k) + m$ [i.e., the scalar part of $\hat{D}_{\rm space}(k) + m$]. The linear maps $\hat{\mathcal{U}}^{(r)}(N_\beta/2)$ on $W \oplus W$ and $\hat{\mathcal{V}}(N_\beta)$ on W in (12) and (13) are defined as follows. Exploiting the periodicity of the link variables to define $d_j^{(r)}(k)$ for all $k \in \mathbf{Z}$, periodic under $k \to k + N_\beta$, and thereby define $\hat{D}^{(r)}\hat{\Psi}(k)$ for all $k \in \mathbf{Z}$, we consider the equation $\hat{D}^{(r)}\hat{\Psi}(k) = 0$ [no periodicity requirement on $\hat{\Psi}(k)$]. In the $r \neq 1$ case, since the $d_\pm^{(r)}(k)$'s are invertible, it is clear from (11) that solutions $\hat{\Psi}(k)$ are specified by two initial values. Thus, the solution space is isomorphic to $W \oplus W$. Setting $\hat{\Psi}_1(n) = \hat{\Psi}(2n)$ and $\hat{\Psi}_2(n) = \hat{\Psi}(2n+1)$ the solutions are determined from their initial values via an evolution operator: $\binom{\hat{\Psi}_1(n)}{\hat{\Psi}_2(n)} = \hat{\mathcal{U}}^{(r)}(n)\binom{\hat{\Psi}_1(0)}{\hat{\Psi}_2(0)}$. The operator $\hat{\mathcal{U}}^{(r)}(N_\beta/2)$ appearing in (12) can also be characterized as follows: Because of the N_β periodicity of the $d_j^{(r)}(k)$'s in (11) there is a linear map on the solution space defined by $\hat{\Psi}(k) \mapsto \hat{\Psi}(k + N_\beta)$, or, equivalently, $(\hat{\Psi}_1(n), \hat{\Psi}_2(n)) \mapsto (\hat{\Psi}_1(n + N_\beta/2), \hat{\Psi}_2(n + N_\beta/2))$. This map coincides with $\hat{\mathcal{U}}^{(r)}(N_\beta/2)$ when the solution space is identified with $W \oplus W$.

In the $r = 1$ case, solutions $\hat{D}^{(1)}\hat{\Psi}(k) = 0$ are determined by just a single initial value; this is connected with the noninvertibility of $d_{\pm 1}^{(1)}(k)$ and can be seen, e.g., from the expression (16) below. Thus the solution space in this case is isomorphic to W. The evolution operator $\hat{\mathcal{V}}(k)$ determines solutions from their initial value through $\hat{\Psi}(k) = \hat{\mathcal{V}}(k)\hat{\Psi}(0)$. The $\hat{\mathcal{V}}(N_\beta)$ in (13) can be alternatively characterized as the linear map on the solution space that maps $\hat{\Psi}(k) \mapsto \hat{\Psi}(k + N_\beta)$.

Finite difference approximations to differential operators in one variable and their determinants have been studied in [7,8] and we are going to use a convergence result from there. In the setting of [7,8], specializing to first order differential operator, the operator L and its finite difference approximation $\hat{L}$ have the forms

$$L = L_1(\tau)\frac{d}{d\tau} + L_0(\tau), \qquad \hat{L} = \hat{L}_1(k)\frac{1}{a}\partial + \hat{L}_0(k) \quad (14)$$

$(\tau \in \mathbf{R}, k \in \mathbf{Z})\partial \in \{\partial^+, \partial^-\}$, $\partial^+\hat{\Psi}(k) = \hat{\Psi}(k+1) - \hat{\Psi}(k)$, $\partial^-\hat{\Psi}(k) = \hat{\Psi}(k) - \hat{\Psi}(k-1)$, with $L_j(\tau), \hat{L}_j(k):W \to W$ being periodic under $\tau \to \tau + \beta$, $k \to k + N_\beta$, respectively, and

$$\hat{L}_j(k) = L_j(ka) + O(a) \qquad (j = 0, 1). \qquad (15)$$

Then the solutions to $L\Psi(\tau) = 0$ and $\hat{L}\hat{\Psi}(k) = 0$ are both determined by a single initial value, so the solution spaces in both cases are isomorphic to W. Solutions $\hat{\Psi}$ approximate solutions $\hat{\Psi}$, i.e., if $\hat{\Psi}(0) = \Psi(0)$, then $\hat{\Psi}(k) \approx \Psi(ka)$ for small a. Consequently, the evolution operator $\hat{\mathcal{U}}(k)$ for $\hat{L}\hat{\Psi} = 0$ approximates the evolution operator $\mathcal{U}(\tau)$ for $L\Psi = 0$. [Explicitly, $\mathcal{U}(\tau) = Te^{-\int_0^\tau L_1(t)^{-1}L_0(t)dt}$.] In particular, one has (cf. Sec. 3 of [8]) the *convergence theorem:* $\hat{\mathcal{U}}(N_\beta) \to \mathcal{U}(\beta)$ for $a \to 0$ with $aN_\beta = \beta$ held fixed. An obvious variant of this that we make use of is the following. If p is a multiple of N_β and $\hat{L} = \hat{L}_1\frac{1}{pa}\partial + \hat{L}_0$ with the $\hat{L}_j(k)$'s periodic under $k \to k + N_\beta/p$ and satisfying $\hat{L}_j(k) = L_j(kpa) + O(a)$ $(j = 0, 1)$, then $\hat{U}(N_\beta/p) \to \mathcal{U}(\beta)$ for $a \to 0$. Furthermore, if W is replaced by $W_1 \oplus W_2$ in the preceding, then the convergence theorem continues to hold when ∂ is replaced by $\left(\begin{smallmatrix}\partial & 0\\ 0 & \tilde{\partial}\end{smallmatrix}\right)$ with $\partial, \tilde{\partial} \in \{\partial^+, \partial^-\}$.

In order to apply the convergence theorem to evaluate the $a \to 0$ limits of (12) and (13) we need to rewrite $\hat{D}^{(r)}$ in the form of $\hat{L}$ in (14), or its aforementioned variant. We have been able to do this only in the $r = 0$ and $r = 1$ cases. The problem of evaluating $\lim_{a\to 0}\det D_\alpha^{(r)}$ in the general r case therefore remains for future work; new techniques beyond those of [7,8] may be required for this. In the $r = 1$ case we specialize to a γ representation where $\gamma_4 = \left(\begin{smallmatrix}1 & 0\\ 0 & -1\end{smallmatrix}\right)$ and decompose $W = W_+ \oplus W_-$ so that $\gamma_4 = \pm 1$ on $W_\pm$. Then, in terms of this decomposition,

$$\hat{D}^{(1)} = \hat{L}_1^{(1)}\frac{1}{a}\begin{pmatrix} \partial^- & 0 \\ 0 & \partial^+ \end{pmatrix} + \hat{L}_0^{(1)}, \qquad (16)$$

where

$$\hat{L}_1^{(1)}(k) = \gamma_4\begin{pmatrix} U_4((k-1)a)^{-1} & 0 \\ 0 & U_4(ka) \end{pmatrix},$$

$$\hat{L}_0^{(1)}(k) = \gamma_4 \begin{pmatrix} \frac{1}{a}[1 - U_4((k-1)a)^{-1}] & 0 \\ 0 & \frac{1}{a}[U_4(ka) - 1] \end{pmatrix} + D_{\text{space}}(ka) + m.$$

Clearly $\hat{L}_j^{(1)}(k)$ is periodic under $k \to k + N_\beta$ and

$$\hat{L}_1^{(1)}(k) = \gamma_4 + O(a),$$
$$\hat{L}_0^{(1)}(k) = \gamma_4 A_4(ka) + D_{\text{space}}(ka) + m + O(a).$$

The convergence theorem now gives $\lim_{a\to 0} \hat{\mathcal{V}}(N_\beta) = \mathcal{V}(\beta)$, where $\mathcal{V}(\tau)$, acting on W, is the evolution operator for $D\Psi(\tau) = 0$ with D being the Dirac operator (7) of the continuous time–lattice space setting. Using this and noting $\det[1 + a\hat{M}(k)] = e^{a\mathrm{Tr}\hat{M}(k)} + O(a^2)$, the $a \to 0$ limit of (13) is now obtained:

$$\lim_{a\to 0} \det D_\alpha^{(1)} = \left(\frac{1}{\alpha}\right)^{NN_\beta} e^{-\alpha\beta N/2 + \frac{1}{2}\int_0^\beta \mathrm{Tr}M(\tau)\, d\tau} \times \det[1 - e^{\alpha\beta}\mathcal{V}(\beta)]. \tag{17}$$

The gauge field-independent factor $1/a^{NN_\beta}$, which diverges in the $a \to 0$ limit, is physically inconsequential; it can at most give rise to an overall constant shift in the calculation of certain physical quantities (such as the energy density in finite temperature QCD).

An application of the zeta-regularized determinant formula for differential operators in one variable, Theorem 1 of [7], leads to an expression for $\det_\zeta D_\alpha$ that coincides with (17) without the $1/a^{NN_\beta}$ factor and with $M(\tau)$ replaced by $\pm M(\tau)$ (the details of this are given in [5]). The sign $\pm$ depends on the choice of cut in the complex plane used to define the zeta determinant. Choosing this so that the sign is "+" we then have $\lim_{a\to 0} a^{NN_\beta} \det D_\alpha^{(1)} = \det_\zeta D_\alpha$, which establishes the first part of (10).

In the $r = 0$ case, with $\hat{\Psi}$ represented by $(\hat{\Psi}_1(n), \hat{\Psi}_2(n))$ as before, we have

$$\hat{D}^{(0)} = \hat{L}_1^{(0)} \frac{1}{2a} \begin{pmatrix} \partial^+ & 0 \\ 0 & \partial^- \end{pmatrix} + \hat{L}_0^{(0)}, \tag{18}$$

where

$$\hat{L}_1^{(0)}(n) = \begin{pmatrix} 0 & \gamma_4 U_4((2n-1)a)^{-1} \\ \gamma_4 U_4((2n+1)a) & 0 \end{pmatrix},$$

$$\hat{L}_0^{(0)}(n) = \begin{pmatrix} D_{\text{space}}(2na) + m & \hat{K}_0(n) \\ \hat{J}_0(n) & D_{\text{space}}((2n+1)a) + m \end{pmatrix},$$

$$\hat{J}_0(n) = \gamma_4 \frac{1}{2a}[U_4(2na) - U_4((2n-1)a)^{-1}],$$

$$\hat{K}_0(n) = \gamma_4 \frac{1}{2a}[U_4((2n+1)a) - U_4(2na)^{-1}].$$

Clearly $\hat{L}_j^{(0)}(n)$ is periodic under $n \to n + N_\beta/2$ and

$$\hat{L}_1^{(0)}(n) = \begin{pmatrix} 0 & \gamma_4 \\ \gamma_4 & 0 \end{pmatrix} + O(a),$$
$$\hat{L}_0^{(0)}(n) = \begin{pmatrix} D_{\text{space}}(2na) + m & \gamma_4 A_4(2na) \\ \gamma_4 A_4(2na) & D_{\text{space}}(2na) + m \end{pmatrix} + O(a). \tag{19}$$

The convergence theorem then gives $\lim_{a\to 0} \hat{U}(N_\beta/2) = \mathcal{U}(\beta)$, where $\mathcal{U}(\beta)$, acting on $W \oplus W$, is the evolution operator for $\tilde{D}\binom{\Psi_1}{\Psi_2}(\tau) = 0$ with

$$\tilde{D} = \begin{pmatrix} D_{\text{space}}(\tau) + m & \gamma_4(\frac{d}{d\tau} + A_4(\tau)) \\ \gamma_4(\frac{d}{d\tau} + A_4(\tau)) & D_{\text{space}}(\tau) + m \end{pmatrix}. \tag{20}$$

Introducing $\mathcal{O} = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$ on $W \oplus W$, we find after a little calculation $\mathcal{O}^{-1}\tilde{D}\mathcal{O} = \begin{pmatrix} (\gamma_4\gamma_5)^{-1}D(\gamma_4\gamma_5) & 0 \\ 0 & D \end{pmatrix}$, where D is the Dirac operator (7). It follows that $\mathcal{U}(\beta) = \mathcal{O}^{-1}\begin{pmatrix} (\gamma_4\gamma_5)^{-1}\mathcal{V}(\beta)(\gamma_4\gamma_5) & 0 \\ 0 & \mathcal{V}(\beta) \end{pmatrix}\mathcal{O}$. Using this, the $a \to 0$ limit of (12) is now obtained:

$$\lim_{a\to 0} \det D_\alpha^{(0)} = \left(\frac{1}{2a}\right)^{NN_\beta} e^{-\alpha\beta N} \det[1 - e^{\alpha\beta}\mathcal{V}(\beta)]^2. \tag{21}$$

Again there is a physically inconsequential, divergent factor, $(1/2a)^{NN_\beta}$. Comparing (21) with (17), and noting that $e^{\frac{1}{2}\int_0^\beta \mathrm{Tr}M(\tau)\, d\tau} = e^{m\beta N/2} e^{\frac{1}{2}\int_0^\beta \mathrm{Tr}[\frac{r'}{2a'}\Delta_{\text{space}}(\tau)]\, d\tau}$, where $e^{m\beta N/2}$ is a PIF, we obtain the claimed result (9). Furthermore, an application of Theorem 1 of [7] gives $\det_\zeta(\gamma_4 D_\alpha) = e^{-(1\mp 1)\beta\alpha N/2} \det[1 - e^{\alpha\beta}\mathcal{V}(\beta)]$ where the sign $\mp$ again depends on the choice of cut in the complex plane [5]. It follows that $\lim_{a\to 0}(2a)^{NN_\beta} \det D_\alpha^{(0)} = e^{\pm\beta\alpha N} \det_\zeta(\gamma_4 D_\alpha)^2$. Since $e^{\pm\beta\alpha N}$ is a PIF, this establishes the second part of (10).

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[1] K. G. Wilson, Phys. Rev. D **10**, 2445 (1974).

[2] J. B. Kogut, Phys. Rev. D **11**, 395 (1975); L. Susskind, Phys. Rev. D **16**, 3031 (1977).

[3] N. Kawamoto and J. Smit, Nucl. Phys. **B192**, 100 (1981); H. Kluberg-Stern *et al.*, Nucl. Phys. **B220**, 447 (1983).

[4] For a review see K. Jansen, hep-lat/0311039.

[5] D. H. Adams, "On the QCD Fermion Determinant" (to be published).

[6] P. Gibbs, Phys. Lett. B **172**, 53 (1986).

[7] D. Burghelea, L. Friedlander, and T. Kappeler, Commun. Math. Phys. **138**, 1 (1991).

[8] R. Forman, Commun. Math. Phys. **147**, 485 (1992).