Resonant Excitations of the 't Hooft–Polyakov Monopole

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The spherically symmetric magnetic monopole in an SU(2) gauge theory coupled to a massless Higgs field is shown to possess an infinite number of resonances or quasinormal modes. These modes are eigenfunctions of the isospin 1 perturbation equations with complex eigenvalues, $E_n = \omega_n - i\gamma_n$, satisfying the outgoing radiation condition. For $n \to \infty$, their frequencies ω_n approach the mass of the vector boson, M_W, while their lifetimes $1/\gamma_n$ tend to infinity. The response of the monopole to an arbitrary initial perturbation is largely determined by these resonant modes, whose collective effect leads to the formation of a long living breatherlike excitation with an amplitude decaying at late times as $t^{-5/6}$.

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Magnetic monopoles—magnetically charged, finite energy solutions in field theories with spontaneously broken gauge symmetries [1]—play an important role in a number of field theoretic considerations. They can account for the quantization of the electric charge, catalyze the decay of proton, possibly be among the relevant topological defects in the universe, determine supersymmetric vacua, etc.—see [2] for reviews.

In this Letter we point out yet another interesting aspect of the monopole that seems to have gone unnoticed so far—the existence of its quasinormal modes (QNM) or resonant excitations. We show that the Bogomol'nyi-Prasad-Sommerfield (BPS) monopole admits an infinite number of QNM. They manifest themselves as resonance peaks in the low energy scattering cross section of isospin 1 scalar particles. The QNM can also be described as *complex energy* solutions of the isospin 1 small fluctuation equations that are regular at the origin and satisfy the *outgoing radiation* condition at spatial infinity. For previous studies of the *real energy* small fluctuations around the monopole we refer to [3].

We demonstrate, in particular, that the QNM of the monopole lead to a *universal late time behavior* of the perturbed monopole by giving rise collectively to a quasiperiodic, long living excitation whose amplitude decays as $t^{-5/6}$ at late times (t being the standard Minkowski time). In a recent numerical study of Fodor and Rácz a breatherlike excitation of the monopole retaining a considerable fraction of the energy of the external perturbation has actually been observed [4]. It has been one of our aims to explain their observations.

It is worth mentioning the striking analogy of the $t^{-5/6}$ asymptotic behavior of the monopole with the late time evolution of massive fields in black hole spacetimes [5]. Black holes also possess the QNM (see, e.g., [6] for a recent discussion).

We consider a Yang-Mills–Higgs (YMH) theory with gauge group SU(2) defined by the Lagrangian

$$\mathcal{L} = -\frac{1}{4} F^a_{\mu\nu} F^{a\mu\nu} + \frac{1}{2} D_\mu \Phi^a D^\mu \Phi^a - \frac{\lambda}{4}(\Phi^a\Phi^a - 1)^2. \tag{1}$$

Here $F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + \varepsilon_{abc} A^b_\mu A^c_\nu$ is the non-Abelian field strength tensor, $D_\mu \Phi^a = \partial_\mu \Phi^a + \varepsilon_{abc} A^b_\mu \Phi^c$ denotes the covariant derivative, and in our units the mass of the gauge bosons is equal to one, $M_W = 1$, while the mass of the Higgs particle is $\sqrt{2\lambda}$. The energy-momentum tensor of the theory defined by Eq. (1) is $T^\mu_\nu = -F^a_{\nu\rho} F^{a\mu\rho} + D^\mu \Phi^a D_\nu \Phi^a - \delta^\mu_\nu \mathcal{L}$.

We restrict our analysis to the "minimal" spherically symmetric sector, where the ansatz for the YMH fields is given by $A^a_0 = 0$,

$$A^a_i = \varepsilon_{aik} \frac{x^k}{r^2}[1 - W(t,r)], \qquad \Phi^a = \frac{x^a}{r^2} H(t,r), \tag{2}$$

where $a, i, k = 1, 2, 3$ and $r^2 = x^k x^k$. With $\Box \equiv \frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial r^2}$ the YMH equations reduce to

$$\begin{aligned} &(r^2\Box + W^2 + H^2 - 1)W = 0, \\ &[r^2\Box + 2W^2 + \lambda(H^2 - r^2)]H = 0. \end{aligned} \tag{3}$$

The static, finite energy solution of these equations is the 't Hooft–Polyakov monopole [1]. For the special case $\lambda = 0$ the solution is known analytically—the BPS monopole,

$$W(r) = \frac{r}{\sinh r}, \qquad H(r) = r\coth r - 1. \tag{4}$$

We consider small fluctuations around the static monopole background: $W \to W(r) + w(t,r)$ and $H \to H(r) + \sqrt{2}\, h(t,r)$. Linearizing Eqs. (3) with respect to w and h, we obtain

$$(r^2\Box + 3W^2 + H^2 - 1)w = -q\, 2\sqrt{2} WHh, \tag{5}$$

$$[r^2\Box + 2W^2 + \lambda(3H^2 - r^2)]h = -q\, 2\sqrt{2} WHw. \tag{6}$$

Here an auxiliary parameter, q, has been introduced, presently $q = 1$. In this Letter we concentrate on the case $\lambda = 0$, when W, H are given by Eq. (4).

Resonant scattering.—First we shall demonstrate that Eqs. (5) and (6) describe resonance phenomena indeed. Separating the variables as $w = \mathrm{Re}(e^{-i\omega t} w_\omega(r))$ and $h = \mathrm{Re}(e^{-i\omega t} h_\omega(r))$ with real ω, Eqs. (5) and (6) become a standard two-channel Schrödinger system. As is clear from what follows, the frequency spectrum is continuous, $\omega^2 \geq 0$. Regular solutions of Eqs. (5) and (6) have to satisfy the conditions $w_\omega \sim h_\omega \sim r^2$ for $r \to 0$, and they can be normalized for $r \to \infty$ such that

$$h_\omega(r) \to \sin[\omega r + \delta(\omega)], \quad w_\omega(r) \to C(\omega) r^{1/\nu} e^{-\nu r}, \quad (7)$$

where $\nu = \sqrt{1 - \omega^2}$. The h field is massless, so it oscillates as $r \to \infty$ for any value of ω. The w field is massive, and for $\omega^2 < 1$ it shows bound-state-type behavior, with exponential decay as $r \to \infty$. The fact that the w field is *nonradiative* for $\omega^2 < 1$ plays the crucial role in our analysis, and below we concentrate on this frequency range. Equations (5) and (6) describe in this case the scattering of a massless h radiation on the monopole surrounded by a confined massive w field. This is effectively a *one-channel* scattering problem. The scattering cross section is therefore given by $\sigma(\omega) = (4\pi/\omega^2)\sin^2(\delta(\omega))$. As the interaction of the h field with the monopole is, in fact, *short range*, $\sigma(\omega)$ is finite. We integrate Eqs. (5) and (6) numerically to obtain $w_\omega(r)$ and $h_\omega(r)$ subject to the boundary conditions (7). The resulting cross section $\sigma(\omega)$ shown in Fig. 1 exhibits a sequence of resonant peaks accumulating near the value $\omega = 1$. This can be so interpreted that for certain energies of the incident h radiation the monopole core gets strongly excited.

QNM—numerical results.—The scattering resonances can usually be related to the quasinormal modes—complex energy solutions satisfying the purely outgoing wave condition at $r = \infty$. To construct the QNM, we integrate Eqs. (5) and (6) with $w = \mathrm{Re}(e^{-iEt} w_E(r))$ and $h = \mathrm{Re}(e^{-iEt} h_E(r))$, where w_E and h_E are complex, the energy $E = \omega - i\gamma$, and

$$Ar^2 \leftarrow h_E(r) \to e^{iEr}, \quad Br^2 \leftarrow w_E(r) \to C r^{1/\nu} e^{-\nu r}, \quad (8)$$

for $0 \leftarrow r \to \infty$. Here A, B, C are complex constants. With the "shooting to a fitting point" numerical method we find a discrete family of global solutions $w_E(r)$, $h_E(r)$ subject to the boundary conditions (8) labeled by $n = 1, 2, \ldots$, the number of nodes of $\mathrm{Im}(w_E(r))$ (see Fig. 2). Notice that $h_E \sim e^{i\omega r + \gamma r}$ *grows* at infinity—the QNM are not physical solutions themselves, but only approximate such solutions for a fixed r and for $t \to \infty$. The first 10 eigenvalues $E_n = \omega_n - i\gamma_n$ are listed in Table I.

Table I clearly indicates that $\omega_n \to 1$ and $\gamma_n \to 0$ for growing n. It seems that the QNM can be obtained for any n, thus composing an infinite family. The values of ω_n coincide well with the positions of the resonance peaks shown in Fig. 1.

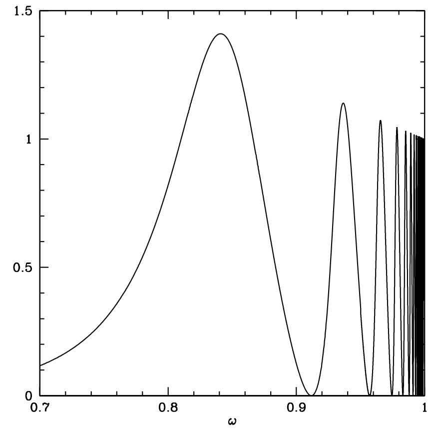

FIG. 1. The scattering cross section $\sigma(\omega)$.

QNM—qualitative analysis.—The existence of the QNM of the BPS monopole can be qualitatively understood as follows. Let us consider q introduced in Eqs. (5) and (6) as a free parameter, $q \in [0, 1]$, and denote the corresponding solutions by $w_{(q)}$ and $h_{(q)}$. For $q = 0$ the equations decouple. Setting $h_{(0)}(t, r) = e^{-i\omega t}(C_+ h_+ + C_- h_-)$ with constant $C_\pm$, Eq. (6) is then solved by

$$h_\pm(r) = (\coth r \mp i\omega) e^{\pm i\omega r}. \quad (9)$$

Equation (5) with $w_{(0)}(t, r) = e^{-i\omega t} w(r)$ reduces to the eigenvalue problem

$$\left(-\frac{d^2}{dr^2} + \frac{3W^2 + H^2 - 1}{r^2}\right) w(r) = \omega^2 w(r). \quad (10)$$

As the potential in this equation has an attractive Coulombian tail, since it behaves as $1 - 2/r + O(e^{-r})$ for $r \to \infty$, there are infinitely many bound states, $w(r) = w_n(r)$ for $\omega^2 = \omega_n^2$, $n = 1, 2, \ldots$. Several low lying ω_n's are 0.798, 0.926, 0.961, 0.984, 0.995 for $n = 1, 2, 3, 5, 10$, respectively. The nth eigenfunction $w_n(r)$ has $n - 1$ nodes in the interval $r \in [0, \infty)$ and can be normalized by the

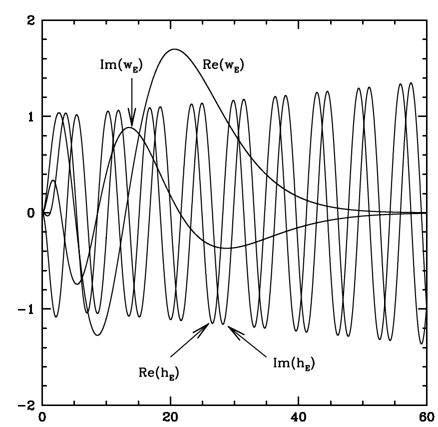

FIG. 2. The complex solutions of Eqs. (5) and (6) for $n = 3$.

TABLE I. The eigenvalues of the first ten QNM.

n	$\omega_n - i\gamma_n$	n	$\omega_n - i\gamma_n$
1	$0.8473 - i\,0.5077 \times 10^{-1}$	6	$0.9888 - i\,0.8396 \times 10^{-3}$
2	$0.9332 - i\,0.1384 \times 10^{-1}$	7	$0.9915 - i\,0.5499 \times 10^{-3}$
3	$0.9637 - i\,0.5218 \times 10^{-2}$	8	$0.9933 - i\,0.3794 \times 10^{-3}$
4	$0.9774 - i\,0.2488 \times 10^{-2}$	9	$0.9946 - i\,0.2731 \times 10^{-3}$
5	$0.9846 - i\,0.1375 \times 10^{-2}$	10	$0.9956 - i\,0.2030 \times 10^{-3}$

condition $\int_0^\infty w_n^2 dr = 1$. For large n the $w_n(r)$'s extend to the asymptotic region where the potential is Coulombian. As a result, they can be well approximated by solutions of the hydrogen atom problem. This implies that for $n \to \infty$ one has $\omega_n^2 = 1 - 1/n^2 + O(n^{-3})$ and also that $w_n \sim n^{-3/2}$. More precisely, for $\omega = 1$ Eq. (10) admits a limiting solution with infinitely many nodes, $w_\infty(r)$, which itself is not normalizable, but which determines the pointwise limit of $w_n(r)$; i.e., for a *fixed* $\bar{r}$

$$\lim_{n\to\infty} w_n(r) = n^{-3/2} w_\infty(r) \quad \forall\ r < \bar{r}. \tag{11}$$

Let us consider a solution of Eqs. (5) and (6) given by $h_{(0)} = 0$ and $w_{(0)} = \mathrm{Re}(A_n e^{-i\omega_n t} w_n(r))$, where A_n is a constant. "Switching on" a small value of the coupling between the channels, $q \ll 1$, the w-bound state will start losing its energy to the h channel, where this energy will be radiated to infinity. To find approximatively the corresponding solutions $h_{(q)}$ and $w_{(q)}$ by successive iterations, we solve first Eq. (6) for $h_{(q)}$ by replacing w by $w_{(0)}$ in its right-hand side. The solution is regular at the origin and reduces to an outgoing wave at infinity: $h_{(q)} \sim qA_n e^{i\omega_n(r-t)}$ as $r \to \infty$.

Since there is now an outgoing flux of the h radiation, the energy of the w-bound state will be slowly decreasing. In the adiabatic approximation this process is described by a decrease of the amplitude of the w field by replacing $A_n \to A_n(t)$. To determine $A_n(t)$ we use the law of energy conservation, $\partial_\mu T_0^\mu = 0$, whose integral form is

$$\frac{d}{dt}\left(\int_0^\infty r^2 T_0^0\, dr\right) = -\lim_{r\to\infty} r^2 T_0^r. \tag{12}$$

Expanding T_ν^μ up to terms quadratic in w and h, the expression on the left in Eq. (12) is proportional to $\frac{d}{dt}A_n^2$ and determines the decrease of the bound state energy. The expression on the right is proportional to A_n^2 and gives the energy flux at infinity. Thus $\dot{A}_n = -\gamma_n A_n$, from which $A_n(t) = c_n e^{-\gamma_n t}$ with a constant c_n. The coefficient γ_n is given by

$$\gamma_n = q^2 \frac{4}{\omega_n^2(1+\omega_n^2)}\left(\int_0^\infty \frac{h_+ - h_-}{2ir^2}\, WHw_n\, dr\right)^2. \tag{13}$$

Summarizing, upon switching on a small value of q, the stationary bound state of the w field, $w_n(r)$, becomes quasistationary and is approximately given by

$$w_{(q)} = \mathrm{Re}(e^{-iE_n t} c_n w_n), \quad h_{(q)} = q\mathrm{Re}(e^{-iE_n t} c_n h_n), \tag{14}$$

where $E_n = \omega_n - i\gamma_n$, and $h_n(r)$ can be expressed by quadratures in terms of $w_n(r)$. Evaluating the integral in Eq. (13) shows that γ_n decreases as n grows; for example, $\gamma_n/q^2 = 0.057, 0.010, 0.0035, 0.0009, 0.0001$ for $n = 1, 2, 3, 5, 10$, respectively. This can be understood qualitatively: since for large n the coupling $WHw_n \sim n^{-3/2}$ between the w and h channels is small, the decay of the w-bound state to the h channel is unlikely. Using (11) in Eq. (13) yields $\gamma_n \sim n^{-3}$ for $n \to \infty$.

The approximative formulas derived above under the assumption $q \ll 1$ give, in fact, when straightforwardly extrapolated to $q = 1$, a good approximation for $\omega_n - i\gamma_n$. This approximation actually gets better with growing n. We can therefore use the asymptotic relations derived above to obtain for $n \gg 1$

$$1 - \omega_n^2 = n^{-2} + O(n^{-3}), \qquad \gamma_n = bn^{-3}. \tag{15}$$

Using the results of Table I, $b \approx 0.2$.

Collective effect of the QNM.—The numerical simulations in Ref. [4] of the temporal dynamics of the strongly perturbed BPS monopole have shown that a considerable fraction of the energy received by the monopole is not radiated away immediately. It gets "trapped" by the monopole and forms a long living excitation that radiates very slowly, and whose late time behavior is to a large extent independent of the structure of the initial perturbation pulse. We can now offer an explanation to these observations.

It is intuitively clear that, according to the eigenfunctions of the linearized problem, there is a "radiative" sector containing the massless h modes with $\omega^2 > 0$, massive w modes with $\omega^2 > 1$, and also a "nonradiative" sector consisting of the w modes with $\omega^2 < 1$. One expects that a part of the energy received by the monopole will be distributed among the radiative modes and will be radiated away. However, as a generic perturbation will have an overlap also with the nonradiative modes, the remaining energy will get trapped, forming a long living excitation that will decay only due to a slow energy leakage to the radiative channels. In the terminology of black hole physics, the perturbed monopole will keep some of its "hair" for a long time.

Although initially the dynamics will be nonlinear, one expects linear effects to dominate at late times, when a sufficient amount of the received energy is radiated away. Let $t = 0$ be the starting point of the linear regime, when the perturbed monopole is described by $w(0, r) = \delta W(r)$ and $h(0, r) = \delta H(r)$. The subsequent temporal evolution is determined to a large extent by the QNM, since they hold their energies for a long time. Using (14), we therefore approximate the general solution for $t > 0$ by

$$w(t, r) = \mathrm{Re}\left(\sum_{n=1}^\infty c_n e^{-i\omega_n t - \gamma_n t} w_n(r)\right), \tag{16}$$

and similarly for $h(t, r)$. Here $c_n = \int_0^\infty w_n(r)\, \delta W(r) dr$.

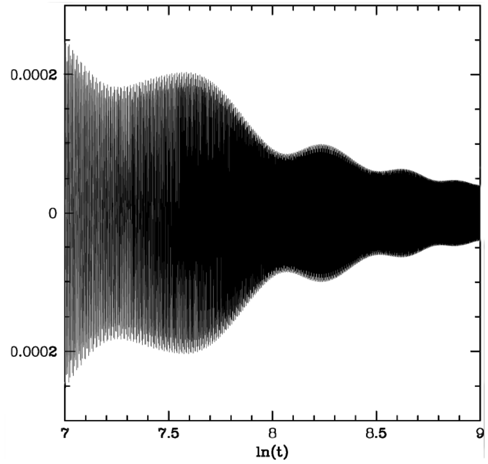

FIG. 3. The profile of $w(t, 1)$ in Eq. (16) corresponding to the initial steplike perturbation, $\delta W(r) = \theta(1 - r)$.

For a localized δW the overlap coefficients c_n are maximal for small n. Therefore, terms with small n dominate at first the sum. They soon are damped, however, since their damping rates, γ_n, are the largest, and so terms with higher n become more important. This "dying out" of modes has indeed been observed in Fig. 6 of [4], and the frequencies ω_n measured there agree well with the values in our Table I.

For any t there is a number, $k(t)$, determined by the condition $\gamma_k t \sim 1$, such that all terms with $n \ll k$ in (16) are already damped, while those with $n \gg k$ are not yet important, since their c_n's are small compared to c_k. The sum therefore is dominated by terms with $n \sim k(t)$. Considering for simplicity only two of them, with frequencies ω_k, ω_{k+1}, their sum gives beats with the base frequency $\omega(t) = \frac{1}{2}(\omega_{k+1} + \omega_k)$ whose amplitude is modulated with the frequency $\Omega(t) = \frac{1}{2}(\omega_{k+1} - \omega_k)$. This explains qualitatively the behavior shown in Fig. 3. Since $\gamma_k \sim k^{-3}$ for large k, it follows that $k(t) \sim t^{1/3}$ for large t. Using (15), we conclude that for large t one has $1 - \omega^2(t) \sim t^{-2/3}$, which explains the feature observed in Fig. 5 of [4]. Similarly, $\Omega(t) \sim t^{-1}$.

Let us determine the late time behavior of the sum (16). Since for large t only terms with $n \geq k(t) \sim t^{1/3}$ contribute to the sum, one can replace ω_n, γ_n, c_n, $w_n(r)$ in (16) by their asymptotic expressions for large n. One has from (15) $\omega_n \approx 1 - \frac{1}{2}n^{-2}$ and $\gamma_n = bn^{-3}$. According to (11), $w_n(r) \approx n^{-3/2} w_\infty(r)$ for $r \leq \bar{r}$, where for $\bar{r}$ one can choose the characteristic size of the wave function with $n = k(t)$, $\bar{r} = \int_0^\infty r w_{k(t)}^2 dr \sim k(t)^2 \sim t^{2/3}$. For a localized δW the overlap coefficients are then given by $c_n = \int_0^\infty w_n \, \delta W dr \approx \mathcal{N} n^{-3/2}$ with $\mathcal{N} = \int_0^\infty w_\infty \delta W dr$. Using all this, the sum in (16) reduces for $t \to \infty$ and for $r < t^{2/3}$ to $w(t, r) \approx \mathcal{N} w_\infty(r) \mathrm{Re}(G(t))$, with

$$G(t) = e^{-it} \sum_{n > t^{1/3}}^{\infty} \frac{1}{n^3} \exp\left(\frac{it}{2n^2} - \frac{bt}{n^3}\right). \qquad (17)$$

This shows that the late time dynamics is indeed *universal,* since changing the initial conditions affects only the normalization $\mathcal{N}$. The final task is to determine the asymptotic behavior of $G(t)$. Transforming the sum (17) to a contour integral and using the saddle point method, we find that $|G(t)| \sim t^{-5/6}$ for $t \to \infty$. This $t^{-5/6}$ exponent explains the feature observed in Fig. 5 of Ref. [4].

The perturbed monopole thus ends up in a long living breathing state dominated by the confined, slowly radiating massive modes of the gauge field. In the region $r \leq t^{2/3}$ this state is characterized by modulated pulsations whose base frequency approaches the vector boson mass, while the amplitude decreases as $t^{-5/6}$.

The total energy of the breather can be obtained by summing over the QNM, $E = \sum_n c_n^2 \omega_n^2 e^{-2\gamma_n t} \sim \sum_n n^{-3} \exp(-2bn^{-3}t) \sim t^{-2/3}$. This includes the energy of the confined massive w modes and also that of the massless h radiation emitted by these modes. E decreases, since there is a flux of the h radiation at infinity, $S = \dot{E} \sim t^{-5/3}$. This exponent agrees with the result of [4], where Fig. 2 shows the flux $S \sim T^{-5/3}$ at future null infinity, with $T \sim t - r$. By continuity, the flux through a 2-sphere of a very large but finite radius r is still $S \sim T^{-5/3}$. But since t is finite then, for $t \gg r$ one has $T \sim t$, which agrees with our result $S \sim t^{-5/3}$.

Although we have considered above only the BPS monopole, similar resonances also exist for a nonzero Higgs self-coupling λ. This follows from the fact that the potential in Eq. (10) is then $1 - 1/r^2 + o(r^{-2})$ for $r \to \infty$, implying the existence of infinitely many bound states.

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