What Does a Strongly Excited 't Hooft–Polyakov Magnetic Monopole Do?

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The time evolution of strongly excited SU(2) Bogomol'nyi-Prasad-Sommerfield magnetic monopoles in Minkowski spacetime is investigated by using numerical simulations based on the technique of conformal compactification and on the use of the hyperboloidal initial value problem. It is found that an initially static monopole does not radiate the entire energy of the exciting pulse toward future null infinity. Rather, a long-lasting quasistable "breathing state" develops in the central region and certain expanding shell structures—built up by very high frequency oscillations—are formed in the far away region.

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Topological solitons play a fundamental role in various field theoretical considerations as they are present in all gauge theory models unifying the strong, weak, and electromagnetic interactions (see, e.g., [1] for a recent review). An important example is the magnetic monopole of 't Hooft–Polyakov [2], which at large distances looks like a Dirac monopole but which is everywhere a regular finite energy solution.

This Letter is to report the results of investigations concerning the time evolution of a strongly excited spherically symmetric SU(2) Bogomol'nyi-Prasad-Sommerfield (BPS) magnetic monopole [3] on a fixed Minkowski background spacetime by using numerical techniques. In particular, the underlying Yang-Mills–Higgs (YMH) system is chosen so that the Yang-Mills field is massive while the Higgs field is massless. The dynamics starts by hitting the static monopole by a concentrated pulse. First, the original pulse splits into two, one directly outgoing and another one going through the origin. Both pulses travel along null geodesics taking away part of the energy of the excitation toward future null infinity, $\mathscr{I}^+$, with the help of the massless Higgs field. It is found, however, that this way the excited monopole releases only about half of the energy received. The rest of the energy of the original pulse seems to be restrained by the monopole in accordance with which it develops a long-lasting quasistable breathing state in the central region and certain expanding shell structures in the far away region. The frequency characterizing the breathing state varies in time, and it approaches asymptotically the value of the vector boson mass from below. In the far away region, where the Yang-Mills and the Higgs fields are practically decoupled, the massless Higgs field does the boring job of transporting the energy released gradually by the central monopole to $\mathscr{I}^+$, while the behavior of the massive Yang-Mills field in the far away asymptotic region can be characterized by the formation of certain expanding shell structures where all the shells are built up by very high frequency oscillations. These oscillations are found to be modulated by the product of a simple time decaying factor of the form $t^{-1/2}$ and of an essentially self-similar expansion.

The time decay of the examined quantities shows a certain type of *universality.* The total energy associated with the hyperboloidal hypersurfaces decreases in time with power $-2/3$, while the amplitude of the oscillating fields decay with power $-5/6$. In a recent work by Forgács and Volkov [4], based on the use of a linear approximation of the BPS monopole, explanation is provided for these universalities.

The investigated dynamical magnetic monopole is described as a coupled SU(2) YMH system. The Yang-Mills field is represented by an su(2)-valued vector potential A_a and the associated two-form field F_{ab} reads as $F_{ab} = \nabla_a A_b - \nabla_b A_a + ig[A_a, A_b]$ where $[\,,\,]$ denotes the product in su(2) and g stands for the gauge coupling constant. The Higgs field (in the adjoint representation) is given by an su(2)-valued function ψ while its gauge covariant derivative reads as $\mathcal{D}_a\psi = \nabla_a\psi + ig[A_a, \psi]$. The dynamics of the investigated YMH system is determined by the Lagrangian $\mathcal{L} = \mathrm{Tr}(F_{ef}F^{ef}) + 2\mathrm{Tr}(\mathcal{D}_e\psi\mathcal{D}^e\psi)$.

Our considerations were restricted to the "minimal" spherically symmetric generalization of the static BPS monopole [2] (see also [5]). Accordingly, the evolution took place on Minkowski spacetime, the line element of which, in spherical coordinates (t, r, θ, ϕ), is $ds^2 = dt^2 - dr^2 - r^2(d\theta^2 + \sin^2\theta\, d\phi^2)$, while the Yang-Mills and Higgs fields, in the so-called *Abelian gauge*, were assumed to possess the form $A_a = -\frac{1}{g}[w\{\tau_2(d\theta)_a - \tau_1 \sin\theta(d\phi)_a\} + \tau_3 \cos\theta(d\phi)_a]$ and $\psi = \tilde{H}\tau_3$, where the generators $\{\tau_I\}$ $(I = 1, 2, 3)$ of su(2) are related to the Pauli matrices σ_I as $\tau_I = \frac{1}{2}\sigma_I$; moreover, w and H were assumed to be smooth functions of t and r.

The field equations relevant for this system are

$$r^2\partial_r^2 w - r^2\partial_t^2 w = w[(w^2 - 1) + g^2r^2H^2], \qquad (1)$$

$$r^2\partial_r^2 H + 2r\partial_r H - r^2\partial_t^2 H = 2w^2H. \qquad (2)$$

The only known analytic solution to (1) and (2) is the static BPS monopole [3]

$$w_0 = \frac{gCr}{\sinh(gCr)}, \qquad H_0 = C\left[\frac{1}{\tanh(gCr)} - \frac{1}{gCr}\right], \quad (3)$$

where C is an arbitrary positive constant. All the results below concern the complete nonlinear evolution of a system yielded by strong impulse-type excitations of this monopole.

The only scale parameter of the above-described system is the vector boson mass $m_w = gH_\infty$, where the limit value $H_\infty = \lim_{r\to\infty} H$ of the Higgs field can in general be shown to be time independent [6]. Since in the case considered here $H_\infty = C \neq 0$, without loss of generality, the parameter choice $g = H_\infty = m_w = 1$ can be ensured to be satisfied by making use of standard rescalings.

To have a computational grid covering the full physical spacetime—ensuring thereby that the outer grid boundary will not have an effect on the time evolution—the technique of conformal compactification, along with the hyperboloidal initial value problem, was used. This way it was possible to study the asymptotic behavior of the fields close to future null infinity, as well as the inner region for considerably long physical time intervals.

The conformal transformation we used is a slight modification of the static hyperboloidal conformal transformation applied by Moncrief [7]. It is defined by introducing first the new coordinates T and R instead of t and r as

$$T = \omega t - \sqrt{\omega^2 r^2 + 1} \quad \text{and} \quad R = \frac{\sqrt{\omega^2 r^2 + 1} - 1}{\omega r}, \quad (4)$$

where $\omega \geq 0$ is an arbitrary constant. The Minkowski spacetime is covered by the coordinate domain given by the inequalities $-\infty < T < +\infty$ and $0 \leq R < 1$. Then the conformally rescaled metric can be given as $\tilde{g}_{ab} = \Omega^2 g_{ab}$, where the conformal factor is $\Omega = \omega(1 - R^2)/2$. The $R = 1$ line represents $\mathscr{I}^+$ through which the conformally rescaled metric $\tilde{g}_{ab}$ smoothly extends to the coordinate domain with $R > 1$.

Using the substitution $H = h/r + H_\infty$ the field equations (1) and (2) in the new coordinates read as

$$\mathfrak{P}\, w = w[(w^2 - 1) + g^2(h + H_\infty R\Omega^{-1})^2], \qquad (5)$$

$$\mathfrak{P}\, h = 2(h + H_\infty R\Omega^{-1})w^2, \qquad (6)$$

where the differential operator $\mathfrak{P}$ is defined as

$$\mathfrak{P} = \frac{4R^2}{(R^2+1)^2}\left[\frac{\Omega^2}{\omega^2}\partial_R^2 - \partial_T^2 - 2R\partial_R\partial_T - \frac{2\Omega}{\omega(R^2+1)}\partial_T - \frac{\Omega R(R^2+3)}{\omega(R^2+1)}\partial_R\right]. \qquad (7)$$

These equations can be put into the form of a first order strongly hyperbolic system [6]. The initial value problem for such a system is known to be well posed [8]. In particular, we solved this first order system numerically by making use of the "method of line" in a fourth order Runge-Kutta scheme following the recipes proposed by Gustafsson *et al.* [8]. The details related to the numerical approach, including representations of derivatives, and treatment of the grid boundaries are to be published in [6]. The convergence tests justified that our code provides a fourth order representation of the selected evolution equations. Moreover, the monitoring of the energy conservation and the preservation of the constraint equations, along with the coincidence between the field values that can be deduced by making use of the Green's function and by the adaptation of our numerical code to the case of massive Klein-Gordon fields, made it apparent that the phenomena described below have to be, in fact, physical properties of the magnetic monopoles.

In each of the numerical simulations, initial data on the $T = 0$ hypersurface was specified for the system of our first order evolution equations. In particular, a superposition of the data associated with the BPS monopole, (3), and of an additional pulse of compact support possessing the form $(\partial_T w)_\circ = c\exp[\frac{d}{(r-a)^2 - b^2}]$ for $r \in [a - b, a + b]$, where $a \geq b > 0$, was used. This corresponds to "hitting" the static monopole between two concentric shells. For the sake of brevity, all the simulations shown below refer to the same pulse corresponding to the choice of the parameters $a = 2$, $b = 1.5$, $c = 70$, $d = 10$, and $\omega = 0.05$. The energy of this pulse is 55.4% of the static monopole. Hence, the yielded dynamical system cannot be considered as being merely a perturbation of the static monopole. We also emphasize that the figures shown below are typical in the sense that for a wide range of the parameters characterizing the exciting pulse the same type of responses are produced by the monopole [6].

In most of the following figures the behavior of various quantities built up from the conserved energy current $j^a = T^a_b(\frac{\partial}{\partial t})^a$ is shown. Consider first the spacetime picture showing the time evolution of $\mathcal{E} = \int j^a n_a \sqrt{|h|}\, d\theta d\phi$, i.e., the energy density associated with shells of radius R, where n_a denotes the future pointed unit normal to the $T =$ const hypersurfaces while h is the determinant of the induced metric on these hypersurfaces (see Fig. 1). In the early part the direct energy transport to $\mathscr{I}^+$ by the Higgs field, with the velocity of light, is apparent. The developing breathing state of the monopole and the formation of the expanding shells of high frequency oscillations are both clearly manifested.

The energy radiated to $\mathscr{I}^+$ can be pictured by plotting (see Fig. 2) the product of the energy current density S and $4\pi r^2$ against time T at $\mathscr{I}^+$ ($R = 1$). It is apparent that the arrival of the two pulses is followed by a small scale but systematic energy loss of the system. From the logarithmic plot the asymptotic behavior $4\pi r^2 S_{\text{asympt}} \approx C_S T^{-\gamma_S}$ can be read off where $\gamma_S \approx 1.66$. In virtue of

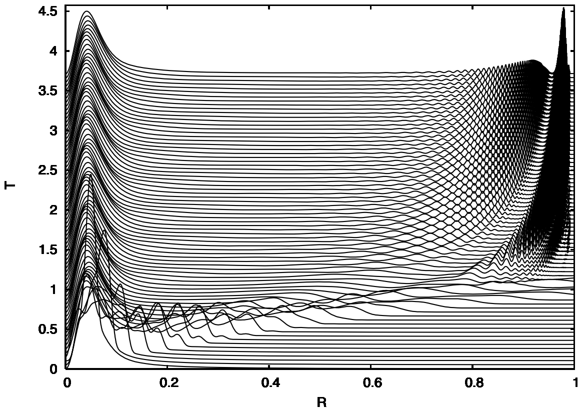

FIG. 1. Spacetime diagram showing the time evolution of the energy density $\mathcal{E}$ associated with shells of radius R; i.e., $E(T) = \int_0^1 \mathcal{E} dR$ gives the total energy of a $T = \text{const}$ hypersurface.

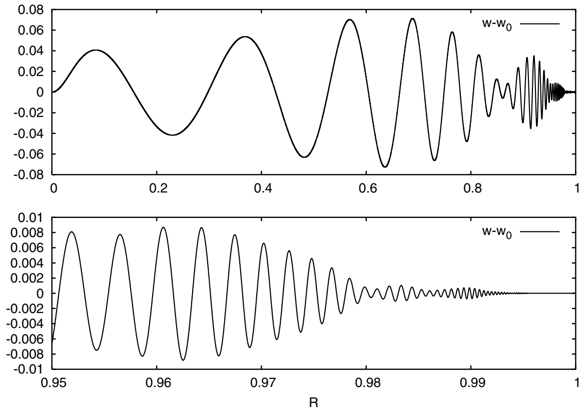

FIG. 3. The difference $w - w_0$ is plotted on the time slice $T = 1.695$.

the energy conservation this relation implies that the energy associated with the $T = \text{const}$ hypersurfaces tends to the energy of the final state as $T^{-2/3}$, in agreement with [4].

By inspection of the evolution of the field variables w and h it is obvious that the expanding high frequency oscillations are associated with the massive Yang-Mills field exclusively. Figure 3 shows a constant time slice of the evolution of $w - w_0$. The formation of the shells built up by high frequency oscillations is transparent on Fig. 3, which is reminiscent of Figs. 1 and 2 of [9]. The behavior of these oscillations in the asymptotic region can be explained by referring to the results of [9] where this phenomenon has already been found to characterize the evolution of the linear massive scalar field. The oscillations are modulated by an overall factor $t^{-1/2}$, scaling them down in time, moreover, by an essentially self-similar expansion, i.e., by a function depending on t and r only in the combination $\rho = r/t$.

Probably, the most interesting unexpected feature of the time evolution is the appearance of the breathing state of the monopole. To have a quantitative characterization of this phenomenon, it is informative to consider a constant R slice of the deviation $\varepsilon - \varepsilon_0$ of the full "energy density" $\varepsilon = j^a n_a$ and that of the static monopole ε_0 (see Fig. 4). Note that the center of the oscillations is actually lower than ε_0, which implies that the time average of the energy contained in the central region is smaller than the energy contained in the same region of the static monopole.

The time dependence of the frequency and the amplitude of the oscillations, shown in Fig. 5, were determined by fitting a simple function of the form $\varepsilon - \varepsilon_0 = a \sin(\omega t + b) + c$ to the numerical data on a sufficiently short time interval so that this interval was shifted point by point through the entire time evolution. The frequency of the oscillations is essentially increasing, asymptotically approaching the value of the vector boson mass m_w. The logarithmic plot suggests the approximate relation $\omega \approx m_w - C_\omega t^{-\gamma_\omega}$, which seems to be valid not merely

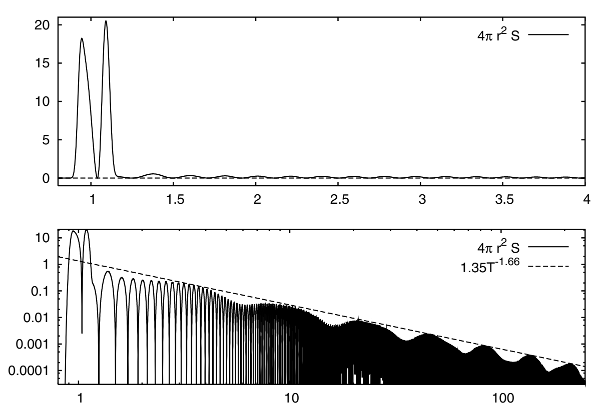

FIG. 2. The time dependence of the product of the energy current density S and $4\pi r^2$ at $\mathscr{I}^+$ is shown on the intervals $0.8 \leq T \leq 4$ and $0.8 \leq T \leq 250$.

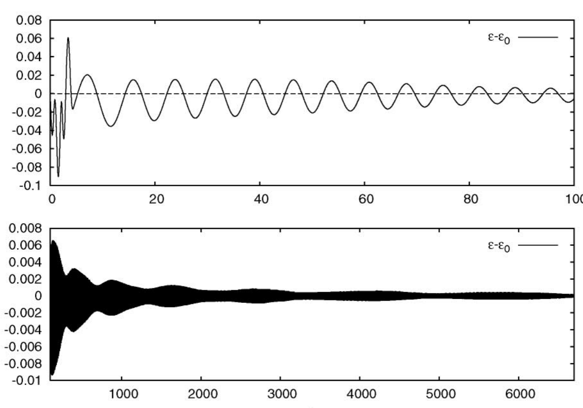

FIG. 4. The deviation $\varepsilon - \varepsilon_0$ is plotted against the physical time intervals $0 \leq t \leq 100$ and $100 \leq t \leq 6700$ at $R = 0.0254$.

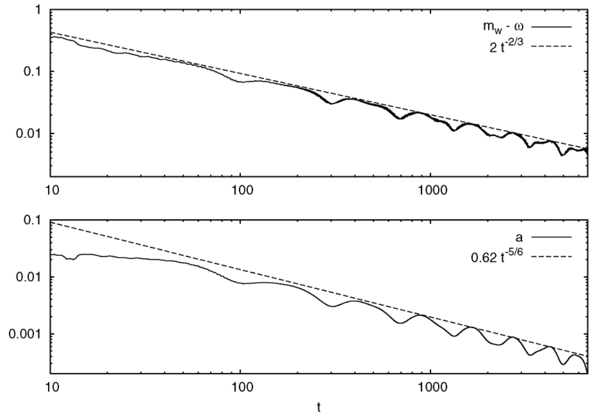

FIG. 5. The frequency difference $m_w - \omega$ and the amplitude a of the oscillations of the breathing state, shown in Fig. 4, is plotted against the physical time t.

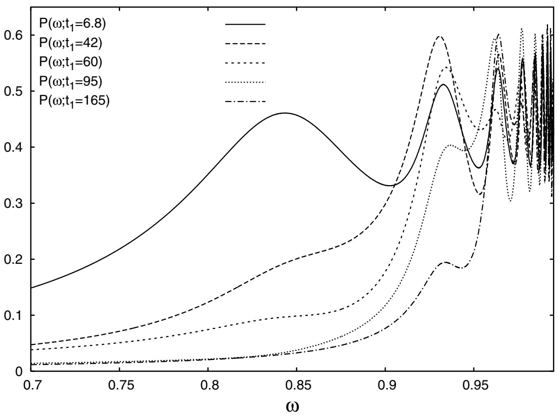

FIG. 6. The power spectrum of the oscillations of $\varepsilon - \varepsilon_0$ at $R = 0.0254$ is plotted for various time intervals $[t_1, t_2]$.

asymptotically but for the entire evolution, where γ_ω was found to a high accuracy to be $2/3$. Exactly the same value $\gamma_\omega = 2/3$ was found in [4]. The asymptotic behavior of the amplitude of the oscillations can be approximated by the simple form $a_{\mathrm{asympt}} \approx C_a t^{-\gamma_a}$, where $\gamma_a \approx 0.833$, which is in good agreement with the value of $-5/6$ of [4]. The energy contained in some finite radius is described by the time average of the energy, i.e., by the function c, not by the amplitude a. For this reason there is no contradiction between the exponents $-5/6$ in the central region and $-2/3$ at infinity. The function c appears to be smaller and decays faster than the amplitude, and can also take positive and negative values depending on the location.

It is also of interest to consider the power spectrum $P(\omega; t_1, t_2)$ of the oscillations. Figure 6 shows $P(\omega; t_1, t_2)$ with t_2 having the fixed value $t_2 = 11\,107$ and t_1 being chosen to take the values 6.8, 42, 60, 95, and 165, respectively. By varying t_1 it is possible to monitor the change of the frequency of the oscillations in the relevant early period. These graphs support the perturbative result of Forgács and Volkov [4] claiming that there has to be an infinite family of "resonant states" possessing discrete frequencies. See, for instance, the peaks of the graph of $P(\omega; t_1 = 6.8)$ at $\omega \approx 0.844, 0.933, 0.9645, 0.9775, 0.985, \ldots$. According to Fig. 6 the lower frequency members of the resonant states die out faster than the higher frequency ones, which is consistent with the above-mentioned increase of the overall frequency of the breathing state.

In summary, in the time evolution of strongly excited BPS monopoles it was found to be generic that a long-lasting quasiperiodic breathing state develops in the central region. The behavior of all the examined quantities justifies the intuitive expectation that in the inner region the system settles down to the original static BPS monopole, while the self-similarly expanding oscillations disperse asymptotically.

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