Berry Phase in a Composite System

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The Berry phase in a composite system with one driven subsystem has been studied in this Letter. We choose two coupled spin-$\frac{1}{2}$ systems as the composite system; one of the subsystems is driven by a time-dependent magnetic field. We show how the Berry phases depend on the coupling between the two subsystems, and the relation between the Berry phases of the composite system and those of its subsystems.

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Geometric phases in quantum theory attracted great interest since Berry [1] showed that the state of a quantum system acquires a purely geometric feature in addition to the usual dynamical phase when it is varied slowly and eventually brought back to its initial form. The Berry phase has been extensively studied [2–4] and generalized in various directions [5–9], such as geometric phases for mixed states [6], for open systems [8], and with a quantized field driving [9]. In a recent paper [10], Sjöqvist calculated the geometric phase for a pair of entangled spins in a *time-independent* uniform magnetic field. This is an interesting development in holonomic quantum computation and it shows how the prior entanglements modify the Berry phase. This study was generalized [11] to the case of spin pairs in a *rotating* magnetic field, which showed that the geometric phase of the whole entangled bipartite system can be decomposed into a sum of geometric phases of the two subsystems, provided the evolution is cyclic.

Entanglement may be created via interactions or jointed measurements, thus the way in which intersubsystem couplings change the Berry phases of a composite system and those of the subsystems is of interest. On the other hand, the Berry phase has very interesting applications, such as the implementation of quantum computation [12–15]; all systems for this purpose are composite, i.e., they consist of at least two subsystems with direct couplings or are coupled via a third party. This again gives rise to questions of how the couplings among the subsystems changes the Berry phase of the composite system and what the relation is between these Berry phases of the composite system and those of the two subsystems.

In this Letter, we investigate the behavior of the Berry phase of two coupled spin-$\frac{1}{2}$ systems, one of the spin-$\frac{1}{2}$ systems is driven by a varying magnetic field. We calculate and analyze the effect of spin-spin coupling on the Berry phase acquired by the composite system and its subsystems.

The Hamiltonian describing a system consisting of two interacting spin-$\frac{1}{2}$ particles in the presence of an external magnetic field takes the form

$$H = \tfrac{1}{2}\alpha\vec{\sigma}_1 \cdot \vec{B}(t) + J(\sigma_1^+\sigma_2^+ + \text{H.c.}), \tag{1}$$

where $\vec{\sigma}_j = (\sigma_j^x, \sigma_j^y, \sigma_j^z)$, σ_j^i are the Pauli operators for subsystem j $(j = 1, 2)$ and $\sigma_j^+ = (1/2)(\sigma_j^x + i\sigma_j^y)$. We will choose $\vec{B}(t) = B_0\hat{n}(t)$ with the unit vector $\hat{n} = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$ and have assumed that only subsystem 1 is driven by the external field. The classical field $\vec{B}(t)$ acts as an external control parameter, as its direction and magnitude can be experimentally altered. J stands for the constant of coupling between the two spin-$\frac{1}{2}$ systems. This coupling is not a typical spin-spin coupling, but rather a toy model describing a double spin flip; nevertheless, the presentation in this Letter may be generalized to the system of nuclear magnetic resonance (NMR) in which quantum computation is implemented [12]. Furthermore, the observation of geometric phase for such a system is feasible by current technology [16].

In a space spanned by $\{|eg\rangle, |ee\rangle, |gg\rangle, |ge\rangle\}$ ($|e\rangle \equiv |\uparrow\rangle$ and $|g\rangle \equiv |\downarrow\rangle$) and in units of $\frac{1}{2}\alpha B_0$, the Hamiltonian Eq. (1) can be written as

$$H = \begin{pmatrix} \cos\theta & 0 & \sin\theta e^{-i\phi} & 0 \\ 0 & \cos\theta & g & \sin\theta e^{-i\phi} \\ \sin\theta e^{i\phi} & g & -\cos\theta & 0 \\ 0 & \sin\theta e^{i\phi} & 0 & -\cos\theta \end{pmatrix}, \tag{2}$$

with $g = \frac{2J}{\alpha B_0}$ a rescaled coupling constant. Keeping θ constant and changing ϕ slowly from 0 to $\phi(T) = 2\pi$ the Berry phase generated after the system undergoing an adiabatic and cyclic evolution starting with an initial state $|\Psi_j(t = 0)\rangle$ may be calculated as follows:

$$\gamma_j = i\int_0^T dt\langle\Psi_j|\dot{\Psi}_j\rangle, \tag{3}$$

where $|\Psi_j\rangle$ $(j = 1, 2, 3, 4)$ are the instantaneous eigenstates of the Hamiltonian Eq. (1) as

$$|\Psi_j\rangle = \frac{1}{\sqrt{M_j}}[a_j(\phi, \theta, g)|eg\rangle + b_j(\phi, \theta, g)|ee\rangle + c_j(\phi, \theta, g)|gg\rangle + d_j(\phi, \theta, g)|ge\rangle], \tag{4}$$

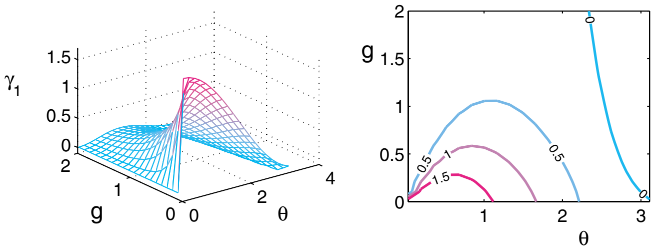

FIG. 1 (color online). Berry phase corresponding to the instantaneous eigenstate $|\Psi_1\rangle$ versus the rescaled coupling constant g and the azimuthal angle θ[Arc]. The plot is presented in units of π for the Berry phase. The right-hand panel is a contour plot for the left-hand panel.

with

$$a_j(\phi, \theta, g) = \sin\theta e^{-i\phi}, \qquad c_j(\phi, \theta, g) = E_j - \cos\theta,$$
$$d_j(\phi, \theta, g) = \frac{g(\cos\theta - E_j)}{\sin\theta - (\cos\theta - E_j)[(\cos\theta + E_j)/\sin\theta]} e^{i\phi},$$
$$b_j(\phi, \theta, g) = -\frac{\cos\theta + E_j}{\sin\theta} e^{-i\phi} d_j(\phi, \theta, g),$$
$$M_j = |a_j|^2 + |b_j|^2 + |c_j|^2 + |d_j|^2, \tag{5}$$

and E_j the corresponding eigenvalues that have the form

$$E_1 = \sqrt{1 + \frac{g^2}{2} + \frac{g}{2}\sqrt{g^2 + 4\sin^2\theta}} = -E_2,$$
$$E_3 = \sqrt{1 + \frac{g^2}{2} - \frac{g}{2}\sqrt{g^2 + 4\sin^2\theta}} = -E_4. \tag{6}$$

In the simplest case where the coupling constant $g = 0$, the eigenvalues $E_\pm = \pm 1$, the corresponding eigenstates follow from Eq. (5) that $a_+ = \sin\theta e^{-i\phi}$, $c_+ = 1 - \cos\theta$, $b_+ = d_+ = 0$, and $a_- = \sin\theta e^{-i\phi}$, $c_- = -1 - \cos\theta$, $b_- = d_- = 0$. These yield the well-known Berry phase $\gamma_+ = \pi(1 + \cos\theta)$ and $\gamma_- = \pi(1 - \cos\theta)$. This result is easy to understand; the subsystem 2 that evolves freely would make no effects on any behaviors of subsystem 1 as long as the whole system is initially prepared in a separable state, hence the Berry phase of the composite system is exactly that of subsystem 1, when subsystem 2 acquires no geometric phase. For a noncyclic and nonadiabatical process, Sjöqvist [10] and Tong [17] draw out

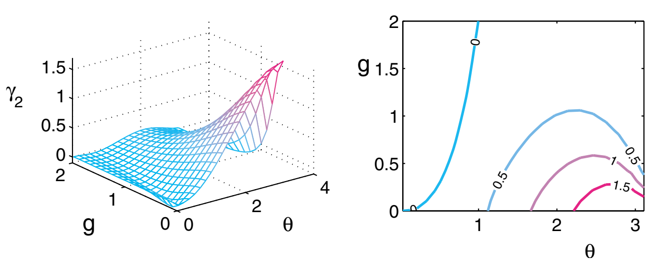

FIG. 2 (color online). The same as Fig. 1, but for instantaneous eigenstate $|\Psi_2\rangle$.

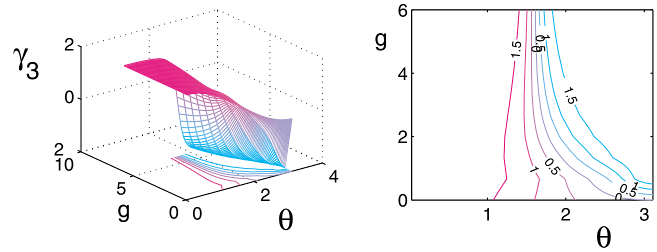

FIG. 3 (color online). Berry phase (in units of π) corresponding to the instantaneous eigenstate $|\Psi_3\rangle$ versus the rescaled coupling constant g and the azimuthal angle θ[Arc]. The right-hand panel is a contour plot for the left-hand panel.

the same results for geometric phases. For the composite system with intersubsystem couplings, the Berry phase is shown in Figs. 1–5 as a function of g and θ, these numerical results are an illustration of Eqs. (3)–(6). Figures 1–4 are for the Berry phases with varying coupling constant g and azimuthal angle θ, whereas Fig. 5 shows the dependence of the Berry phase on the coupling constant with a specific azimuthal angle $\theta = \frac{\pi}{4}$. The common feature of these figures is that with the rescaled coupling constant $g \to \infty$, Berry phases $\gamma_i \to 0$ (all phases are defined modulo 2π throughout this Letter). This limit corresponds to the case when the first term in Hamiltonian Eq. (1) can be ignored. Physically, the spin-spin coupling may modify the azimuthal angle to an effective one with which the system precesses around the z axis. The spin-spin couplings describe a jointed spin flip of the subsystems, the coupling constant then characterizes the flip frequency. Consequently, the effective azimuthal angle should be an average over all possible azimuthal angles which would take positive and negative values with equal probabilities in the limit $g \to \infty$.

From Fig. 1 we see that the Berry phase is a monotonic function of the rescaled coupling constant, while it is maximized for intermediate values of the azimuthal angle θ. The Berry phase for the eigenstate $|\Psi_2\rangle$ has a similar feature as Fig. 2 shows. It is worth noting that $\gamma_1(\theta) = \gamma_2(\pi - \theta)$, this can be easily found by comparing the contour plots in Figs. 1 and 2. This symmetry originates from the Hamiltonian and it is clear that the eigenstate $|\Psi_1\rangle(|\Psi_3\rangle)$ is alternated with $|\Psi_2\rangle(|\Psi_4\rangle)$ when

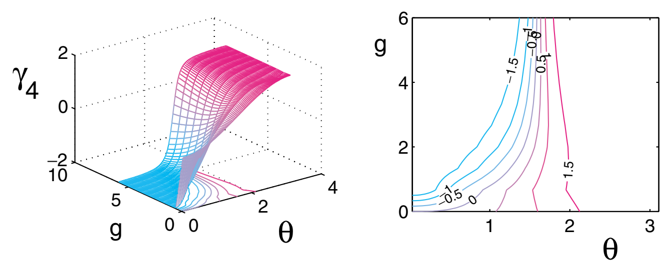

FIG. 4 (color online). The same as Fig. 3, but for the instantaneous eigenstate $|\Psi_4\rangle$.

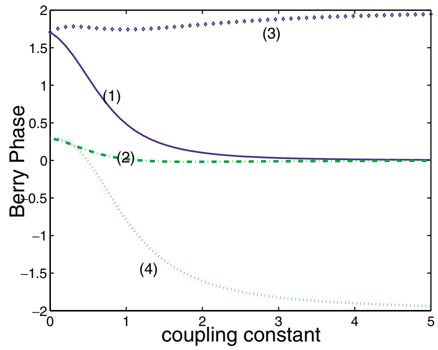

FIG. 5 (color online). The Berry phase versus the rescaled coupling constant g with a specific azimuthal angle $\theta = \frac{\pi}{4}$. The indices on the line indicate the eigenstate by which we obtained the Berry phase.

$\theta \leftrightarrow (\pi - \theta)$, this leads to the symmetry in the Berry phases. The contour plots presented in Figs. 3 and 4 show the same symmetry indeed.

Figure 5 shows the results of Berry phases with a specific azimuthal angle $\theta = \theta_0 = \frac{\pi}{4}$. With $g \to 0$ the Berry phases approach two values (in units of π) of $\gamma_\pm = (1 \pm \cos\theta_0) \simeq 1 \pm 0.707$ as expected, whereas they approach 0 with $g \to \infty$.

Now we are in a position to study the Berry phase of the subsystems and to show what the relation is between these phases. Generally speaking, a state of subsystem is no longer a pure one, so we have to adopt the definition of geometric phase for a mixed state [6]; that is, $\phi_g = \arg\mathrm{Tr}[\rho_0 U(t)]$ with ρ_0 denoting the initial density matrix and $U(t)$ the transport operator which should fulfill the parallel transport evolution condition. This definition is available when the system undergoes a unitary evolution. For the subsystems with nonzero couplings, however, the evolution of each subsystem is not unitary in general. So here we borrow the idea in [7] to define the Berry phase. A nonunitary evolution of a quantal state may be conveniently modeled by attaching an ancilla to the system, in our case the ancilla can always be taken to be the other spin-$\frac{1}{2}$ system. The geometric phase corresponding to this nonunitary evolution is then defined as the geometric phase of the whole system (system + ancilla) that evolves unitarily. For an adiabatic cyclic evolution, this leads to a definition of Berry phase for a mixed state $\rho(t) = \sum_j p_j |E_j(t)\rangle\langle E_j(t)|$,

$$\gamma = \sum_j p_j \gamma_j, \tag{7}$$

where $\gamma_j = i\int_0^T dt\langle E_j(t)|\dot{E}_j(t)\rangle$. The Berry phase Eq. (7) for a mixed state is just an average of the individual Berry phases, weighted by their eigenvalues p_j. To be sure what we have is consistent with known results, we check that this expression reduces to the standard Berry phase $\gamma = i\int_0^T dt\langle\psi(t)|\dot{\psi}(t)\rangle$ for a pure state $\rho(t) = |\psi(t)\rangle\langle\psi(t)|$. In our case we have four density matrices of mixed state for each subsystem that correspond to the four instantaneous eigenstates of the Hamiltonian, respectively. For example, $\rho_1^j(t) = \mathrm{Tr}_2|\Psi_j(t)\rangle\langle\Psi_j(t)|$ represents the jth density matrix for subsystem 1 among the four density matrices, where Tr_2 denotes a trace over subsystem 2. The Berry phase corresponding to the state $\rho_1^j(t)$ is then given by Eq. (7). Actually, the definition Eq. (7) can be derived by the idea of the so-called purifications as follows. We may construct a pure state

$$|\Phi(t)\rangle = \sum_j \sqrt{p_j}\,|E_j(t)\rangle_1 \otimes |j\rangle_a$$

for subsystem 1 + ancilla (for subsystem 2 + ancilla, in the same manner) such that

$$\mathrm{Tr}_a|\Psi(t)\rangle\langle\Psi(t)| = \sum_j p_j|E_j(t)\rangle\langle E_j(t)| = \rho_1(t),$$

where Tr_a denotes a trace over the ancilla and $|E_j(t)\rangle$ represent instantaneous eigenstates of $\rho_1(t)$. Since the states of the ancilla remain unchanged during the evolution, the Berry phase of subsystem 1 is then the Berry phase of the compound (subsystem + ancilla), which yields the definition Eq. (7). The Berry phase of the composite system and those of the two subsystems are illustrated in Fig. 6, a sum of the subsystem's Berry phase is also shown. There is evidence that the Berry phase of the composite system can be decomposed into a sum of the subsystem's Berry phases; it reveals the relation between geometric phases of entangled bipartite systems and those of their subsystems. We can prove this point indeed by expanding the instantaneous eigenstate of the composite system via Schmidt decomposition,

$$|\Psi\rangle = \sum_i \sqrt{p_i}\,|e_i(t)\rangle_1 \otimes |E_i(t)\rangle_2, \tag{8}$$

where $|\Psi\rangle$ denotes one of the instantaneous eigenstates Eq. (4). This expansion yields the reduced density operator $\rho_1(t) = \sum_i p_i|e_i(t)\rangle\langle e_i(t)|$ and $\rho_2(t) = \sum_i p_i|E_i(t)\rangle\langle E_i(t)|$ for subsystems 1 and 2, respectively.

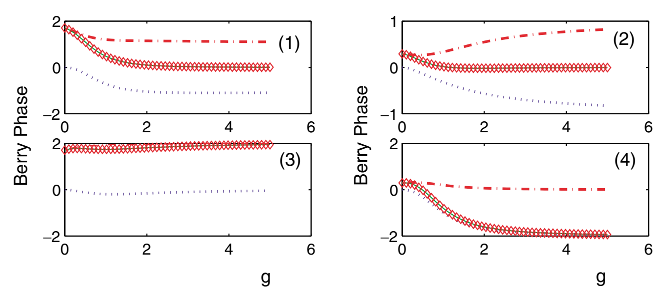

FIG. 6 (color online). The Berry phase for the composite system (diamonds) and for subsystem 1 (dash-dotted line) and subsystem 2 (dotted line). A sum of the subsystem's Berry phase is also presented in the figure (solid line, overlapped by the diamond line). The azimuthal angle $\theta = \frac{\pi}{4}$ was chosen for this plot.

By the definition Eq. (3), the Berry phase corresponding to $|\Psi\rangle$ follows

$$\gamma = i\int_0^T \sum_j p_j \langle e_j(t)|\dot{e}_j(t)\rangle dt + i\int_0^T \sum_j p_j \langle E_j(t)|\dot{E}_j(t)\rangle dt; \tag{9}$$

i.e., the Berry phase of the composite system adds up to be that of the composite system. This additivity holds mathematically when the Schmidt decomposition is available with time-independent coefficients p_i. Physically, time-independent coefficients p_i indicate no population transfer among the eigenstates of the reduced density matrix of subsystem (this is what we called adiabaticity for the subsystem in this Letter). Here the result that the Berry phase of the subsystem adds up to be that for the composite system remains unchanged for all two-subsystem compounds when both the compound and the subsystems undergo a cyclic adiabatic evolution.

The observation of this prediction with NMR experiment is within the reach of current technology [16]. For instance, we can use carbon-13 labeled chloroform in d_6 acetone as the sample, in which the single ^{13}C nucleus and the ^1H nucleus play the role of the two spin-$\frac{1}{2}$ particles. The constant of spin-spin coupling $J\sigma_1^z\sigma_2^z$ in this case is $J \simeq (2\pi)214.5$ Hz, and we may control the rescaled coupling constant g by changing the magnitude of the external magnetic field. We would like to address that the interaction between the two spin-$\frac{1}{2}$ particles in our model is not a typical spin-spin coupling as that in NMR, but rather a toy model describing a double spin flip. So, we have to make a mapping when we employ the presentation in NMR system and when all subsystem are driven by the classical field. Finally, we want to discuss the problem of adiabaticity. Our study is based on an adiabatic cyclic evolution of the composite system. For any subsystem, however, the conditions of adiabaticity are not fulfilled in general; in this sense this is not a Berry phase but a geometric phase for the subsystem when the composite system itself is subject to an adiabatic evolution. But this is not the case in this Letter. It is easily to check that the eigenvalues of the reduced density matrix $\rho_1^j(t) = \mathrm{Tr}_2(|\Psi_j(t)\rangle\langle\Psi_j(t)|$ (for any j) are independent of time, which indicates the population on the eigenstate of $\rho_1^j(t)$ remains unchanged [18]while the composite system follows an adiabatic evolution.

In summary, we have theoretically investigated the Berry phase of a composite system and that of their subsystems. The Berry phase for a mixed state to our best knowledge is a new concept. The relation between those phases is also presented and discussed. These results provide us with a new way to control the Berry phase, which thus might aid in finding some applications in quantum computation. We are investigating possible applications of this effects and its connection to other quantum effects in different systems.

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