Magnetic-Field Generation in Kolmogorov Turbulence

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We analyze the initial, kinematic stage of magnetic field evolution in an isotropic and homogeneous turbulent conducting fluid with a rough velocity field, $v(l) \sim l^\alpha$, $\alpha < 1$. This regime is relevant to the problem of magnetic field generation in fluids with small magnetic Prandtl number, i.e., with Ohmic resistivity much larger than viscosity. We propose that the smaller the roughness exponent α, the larger the magnetic Reynolds number that is needed to excite magnetic fluctuations. This implies that numerical or experimental investigations of magnetohydrodynamic turbulence with small Prandtl numbers need to achieve extremely high resolution in order to describe magnetic phenomena adequately.

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Introduction.—In a turbulent highly conducting fluid, magnetic fields may be amplified since the field lines are generally stretched by randomly moving fluid elements in which these lines are frozen [1,2]. Such a mechanism of turbulent dynamo is consistent with simple analytical models, is confirmed numerically, and is expected to work in a variety of astrophysical systems (galaxy clusters, interstellar medium, stars, planets).

Valuable insight into the nature of this process can be gained from considering the so-called kinematic stage of the dynamo, when the magnetic field is amplified from an initially weak "seed" field. As a simple example, consider the case where the resistive scale is smaller than or of the same order as the viscous scale of the fluid. In a turbulent velocity field with Kolmogorov spectrum, the smallest eddies have the highest shearing rate given by $v(l)/l \sim l^{-2/3}$, where l is the size of the eddies, and $v(l) \sim l^{1/3}$ is their turnover velocity. Therefore, at this stage, the magnetic field grows predominantly on small scales; see, e.g., [3].

The magnetic energy collapses toward small, resistive scales during this initial evolution, until the field is strong enough to affect the dynamics of fluid through the Lorentz force. The back reaction occurs when the growing magnetic energy at the resistive scale approaches the kinetic energy of the smallest-scale eddies [4,5]. Such behavior is the evidence of small-scale turbulent dynamo; it is firmly established in numerical experiments, and can be derived analytically [3,6–12]. Since in this example, the resistive scale is smaller than the viscous one, the dynamo is essentially governed by a smooth, viscous-scale velocity field.

In order to study the growth of magnetic energy on larger scales we have to understand how the magnetic field is generated in the inertial interval of the turbulence (the interval of scales much smaller than the external scale where the turbulence is excited, and much larger than the viscous scale where the turbulent energy is dissipated). This problem is nontrivial since in this interval the velocity field is not smooth, i.e., $v(l) \sim l^\alpha$, with $\alpha < 1$. This situation is especially relevant for the case of small magnetic Prandtl number fluids ($\mathrm{Pm} = \nu/\eta$, where ν is the viscosity and η is the resistivity), where the magnetic energy is concentrated mostly in the inertial interval of the velocity field.

This problem was first addressed by Batchelor [6], who used the analogy between magnetic field lines and vorticity lines to conclude that when $\nu \le \eta$, magnetic energy is not amplified. This analogy was criticized in [7] on the grounds that the magnetic field can have arbitrary initial conditions, while the vorticity field is related to the velocity field by $\boldsymbol{\omega} = \nabla \times \mathbf{v}$. It was argued further in [13] that the stretching of magnetic field lines typically dominates over resistive dissipation, thereby making dynamo action possible for $\nu \le \eta$. Direct numerical simulations of MHD turbulence with $\mathrm{Pm} = 1$ indeed confirm that weak magnetic fluctuations are generally amplified by Navier-Stokes velocity fields except for special cases where the initial magnetic field distribution is close to that of the vorticity [14,15].

Recently, a number of high-resolution numerical simulations of MHD turbulence with small magnetic Prandtl numbers appeared in which magnetic fluctuations were not amplified [16–21], which revived the interest in the possibility that dynamo action may not exist for Kolmogorov turbulence with $\mathrm{Pm} \ll 1$; see, e.g., Ref. [20].

This possibility, if correct would contradict the conclusions of [13], and possibly be at odds with the astrophysical observations showing that magnetic fields are efficiently generated in the convection zones of late-type stars and in planetary interiors where the magnetic Prandtl numbers are small (e.g., in the geodynamo, $\mathrm{Pm} \sim 10^{-5}$–10^{-6}, in the solar photosphere, $\mathrm{Pm} \sim 10^{-7}$). Although, to be fair, in the astrophysical cases it is difficult to estimate the impact of (kinetic) helicity and inverse cascades on the small-scale dynamo problem. In any case, the apparent disagreement between theory and numerical simulations has motivated our interest in the problem.

In this Letter we argue that dynamo action is always possible in a rough velocity field, such as the Kolmogorov field. We find, however, that the magnetic Reynolds number, $\mathrm{Rm} = v(L_0)L_0/\eta$, for dynamo action, and the associated numerical resolution required to represent correctly the growing modes strongly depend on the roughness exponent of the velocity field, α; furthermore, the rougher the velocity field, the larger the required resolution. This result explains why dynamo action is hard to achieve in experiments with small magnetic Prandtl numbers, while it is easily achieved when the magnetic Prandtl number is large, $\mathrm{Pm} \geq 1$. In the latter case the velocity field is effectively smooth at the resistive scales, where the magnetic energy is concentrated.

Our analysis is based on the so-called Kazantsev model for kinematic dynamo action, which is exactly solvable and allows velocity roughness exponent α to vary. The model contains a number of simplifying assumptions; however, it proves rather effective to reconcile the analytical and numerical results.

The Kazantsev model for a rough velocity field.— Consider the induction equation for the magnetic field:

$$\partial B^i/\partial t + v^j \partial B^i/\partial x^j - B^j \partial v^i/\partial x^j = \eta \Delta B^i, \qquad (1)$$

where $B^i(\mathbf{x}, t)$ is the magnetic field, $v^i(\mathbf{x}, t)$ is the fluid velocity, η is the (small) Ohmic resistivity, and summation is assumed over repeated indices. At the initial stage of evolution, a weak magnetic field is passively advected by the fluid, and one is justified in introducing simplifying assumptions about the fluid velocity to make the problem manageable. Kazantsev [8] and Kraichnan [22] introduced the model based on the Gaussian, short-time correlated velocity field, with zero mean and the covariance

$$\langle v^i(\mathbf{x}, t) v^j(\mathbf{x}', t')\rangle = \kappa^{ij}(\mathbf{x} - \mathbf{x}')\delta(t - t'). \qquad (2)$$

This model is a valuable tool for the analytical investigation of kinematic dynamos; direct numerical simulations reveal that a purely Gaussian velocity field amplifies small magnetic fluctuations in a similar manner as the true Navier-Stokes field [14].

Assuming isotropy and homogeneity, the velocity correlation function has the form

$$\kappa^{ij}(\mathbf{r}) = \kappa_N(r)\left(\delta^{ij} - \frac{r^i r^j}{r^2}\right) + \kappa_L(r)\frac{r^i r^j}{r^2}, \qquad (3)$$

where $\mathbf{r} = \mathbf{x} - \mathbf{x}'$. If we further assume that the velocity field is incompressible, we have $\kappa_N = \kappa_L + (r\kappa_L')/2$, and velocity statistics can be characterized by the single scalar function, $\kappa_L(r)$.

The model defined by (1)–(3) allows one to write a closed equation for the correlation function of the magnetic field, namely,

$$\langle B^i(\mathbf{x}, t) B^j(\mathbf{x}', t)\rangle = H^{ij}(\mathbf{x} - \mathbf{x}', t), \qquad (4)$$

where, analogously to (3), the H^{ij} function can be represented as

$$H^{ij} = H_N(r, t)\left(\delta^{ij} - \frac{r^i r^j}{r^2}\right) + H_L(r, t)\frac{r^i r^j}{r^2}, \qquad (5)$$

and, furthermore, the condition $\nabla \cdot \mathbf{B} = 0$ gives $H_N = H_L + (rH_L')/2$. We characterize the magnetic field correlator by the function $H_L(r, t)$. The equation for this function can be found by differentiating (4) with respect to time and by using Eqs. (1)–(3). A rather tedious but essentially straightforward calculation gives

$$\partial_t H_L = \kappa H_L'' + \left(\frac{4}{r}\kappa + \kappa'\right)H_L' + \left(\kappa'' + \frac{4}{r}\kappa'\right)H_L, \qquad (6)$$

where we have introduced the "renormalized" velocity correlation function $\kappa(r) = 2\eta + \kappa_L(0) - \kappa_L(r)$, and primes denote the derivatives with respect to r. Equation (6) was originally derived by Kazantsev [8] and can be rewritten in a related form that formally coincides with the Shrödinger equation in imaginary time. Effecting the change of the variable, $H_L = \psi(r, t)r^{-2}\kappa(r)^{-1/2}$, one obtains

$$\partial_t \psi = \kappa(r)\psi'' - V(r)\psi, \qquad (7)$$

which describes the wave function of a quantum particle with variable mass, $m(r) = 1/[2\kappa(r)]$, in a one-dimensional potential ($r > 0$):

$$V(r) = \frac{2}{r^2}\kappa(r) - \frac{1}{2}\kappa''(r) - \frac{2}{r}\kappa'(r) - \frac{(\kappa'(r))^2}{4\kappa(r)}. \qquad (8)$$

Equation (7) can be investigated for different choices of $\kappa(r)$; however, we restrict ourselves to the inertial interval of the turbulence, where the velocity correlator has power-law asymptotics, $\kappa(r) \propto r^\beta$. The exponent β can be found from the scaling of turbulent diffusivity, $D \sim v(r)r \sim r^{1+\alpha}$. Indeed, in the derivation of Eq. (6) we used the integral $D = \int_0^\infty \langle [v(x, t) - v(x', t)] \times [v(x, t') - v(x', t')]\rangle d(t - t') = \kappa_L(0) - \kappa_L(r) \sim r^\beta$, which is the turbulent diffusivity [13]. Comparing the two expressions, we see that $\beta = 1 + \alpha$.

The Schrödinger equation (7) has the effective potential $U_{\mathrm{eff}}(r) = V(r)/\kappa(r) = A(\beta)/r^2$, where $A(\beta) = 2 - 3\beta/2 - 3\beta^2/4$. When $A(\beta) < -1/4$, the quantum particle falls toward the origin [23], and its wave function is concentrated at the smallest, resistive scale. This behavior is the manifestation of the dynamo mechanism that was discussed in the introduction. We obtain that $A(1 + \alpha) < -1/4$ for any roughness exponent of the velocity field, $0 < \alpha < 1$, and, therefore, dynamo action is always possible. The same conclusion was also reached in [13].

Magnetic field correlator and dynamo growth rates.— In order to find the wave function (the magnetic field correlator), boundary conditions must be specified . A small-scale regularization is naturally given by Ohmic resistivity. For scales much smaller than the correlation scale of the velocity field, the κ function can be expanded as $\kappa(r) \simeq \kappa_L(0)(2\eta + r^{1+\alpha})$, which corresponds to the limit of infinitely small Prandtl number. In this expression and in what follows, we use the dimensionless variables η, measured in units of the large-scale turbulent diffusivity, $\kappa_L(0)$, and r, measured in units of the integral scale, L_0.

The boundary condition at the origin follows from finiteness of magnetic energy, $\lim_{r\to 0} H_L(r,t) = H_0(t) < \infty$. The boundary condition at large scales follows from the absence of a mean magnetic field, $\lim_{r\to\infty} H_L(r,t) = 0$. We will see presently that in the kinematic regime the magnetic energy is concentrated at the resistive scale, and the corresponding wave function decays exponentially fast in the inertial interval, $r \gg r_\eta$, so as to become independent of the large-scale properties of the velocity field, as expected.

Problem (6) can be cast into the Sturm-Liouville form by the change of variable $H_L(r,t) = h(r,t)/r^2$, indicating that its solution can be expanded in the eigenfunctions of the corresponding Sturm-Liouville operator. The maximum growth rate is then given by the largest eigenvalue of the operator on the right-hand side of Eqs. (6) or (7), and the correlation function, H_L, corresponds to the ground-state of this operator.

Following Kazantsev [8], we look for the solution of (7) in the form $\psi(r,t) = \varphi(r)\exp(\lambda t)$. In the inertial interval, $2\eta \ll r^\beta$, the equation for the φ function reads

$$-\varphi'' + \left[\frac{-3\beta^2 - 6\beta + 8}{4r^2} + \frac{\lambda}{r^\beta}\right]\varphi = 0, \qquad (9)$$

where λ is dimensionless and is measured in units of $\kappa_L(0)/L_0^2$. The parameter λ must be chosen so as to zero the ground-state energy of the potential in Eq. (9).

For small λ, the potential is dominated by the first term in the square brackets, and one can easily check that the corresponding wave function oscillates in the inertial interval. However, the ground-state wave function cannot oscillate; therefore, the growth rate, λ, must be such that the second term in the square brackets of (9) dominates the first one in the inertial interval. At the resistive scale, $r_\eta = (2\eta)^{1/\beta}$, both terms should be of the same order; therefore, $\lambda \sim \eta^{(\beta-2)/\beta}$.

The wave function corresponding to the growing solution, $\lambda > 0$, decays exponentially fast for $r \gg \eta^{1/\beta}$, $\varphi \propto \exp[-2\sqrt{\lambda} r^{(2-\beta)/2}/(2-\beta)]$. This function is concentrated at the resistive scale, and its growth rate, λ, is of the order of the eddy turnover time at this scale, in agreement with our qualitative discussion in the introduction. When one changes the magnetic Prandtl number (by changing viscosity, for instance), the effective roughness of the velocity field at the resistive scale changes. We therefore propose that the effect of variations in Prandtl number can be studied in terms of Eqs. (6) and (9) with different roughness exponents. By this analogy, the limit of the smooth velocity field, $\beta = 2$, corresponds to large Pm, while the other limit of the Kolmogorov-scaled velocity field, $\beta = 4/3$, corresponds to small Pm.

We note that the rougher the velocity field, the broader the wave function compared to the resistive scale, and, therefore, the larger the magnetic Reynolds number necessary to generate magnetic fluctuations. To estimate the critical resolution for the dynamo onset, we solved Eq. (6) numerically with the large-scale boundary condition, $H_L(\ell) = 0$. For given β and η, we increased the "system size," ℓ, until H_L started to grow, and we thus found the critical value $\ell = \ell_c$. An analogous calculation, leading to the same results, could be done by fixing ℓ and changing η.

To characterize the inertial range, we introduce the resolution parameter, $R = \ell_c/r_\eta = \ell_c/(2\eta)^{1/\beta}$. This parameter is universal, in the sense that it is independent of the large- and small-scale properties of the velocity field. We obtain that when the velocity roughness increases from $\beta = 2.0$ to $\beta = 1.3$, the corresponding resolution parameter increases from $R \approx 3.8$ to 29; see Fig. 1. This means that if we simulated the random Eq. (1) directly, the required numerical resolution would increase by about an order of magnitude in each spatial direction as we go from a smooth velocity to the Kolmogorov-scaled one, i.e., from large to small Prandtl numbers. Practically, the resolution must increase even more since the subresistive scales must be resolved as well.

The behavior in the subresistive scales, $r \ll \eta^{1/\beta}$, can easily be described using Eq. (6). By direct substitution, one can check that the magnetic field correlator can be expanded as $H_L(r,t) = H_0[1 - r^\beta/(2\eta) + \ldots]\exp(\lambda t)$, which implies that the spectrum of the magnetic field decays as $H_k \sim k^2|B_k|^2 \propto k^{-1-\beta}$ in the subresistive region, $\eta^{-1/\beta} \ll k \ll \nu^{-1/\beta}$. With this spectrum, the rate of magnetic energy dissipation at the viscous-scale exceeds that at the resistive scale.

This last result is an artifact of the short-time correlated subresistive eddies, and is not applicable in the Kolmogorov turbulence where the velocity correlation

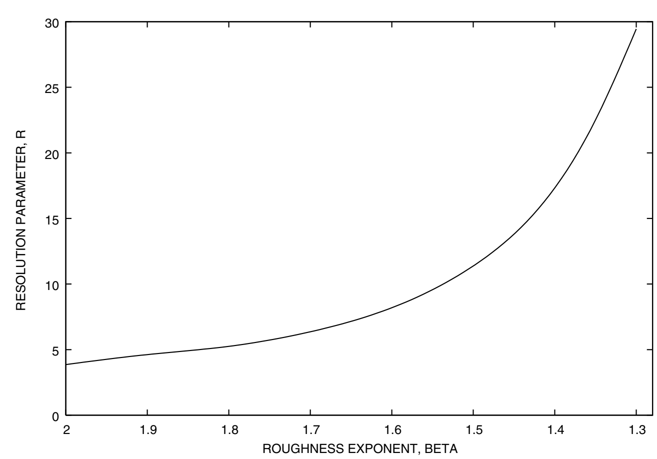

FIG. 1. The numerically computed resolution parameter, $R = \ell_c/r_\eta$, as a function of the velocity roughness exponent, β.

and turnover times are comparable, $\tau_{\rm corr}(l) \sim l/v(l)$. At the subresistive scales, $l < l_\eta$, this correlation time is larger than the resistive relaxation time, $\tau_\eta \sim l^2/\eta$, while in the Kazantsev model it is smaller than τ_η. However, model (2) is physically self-consistent. The large magnetic energy dissipation is balanced by the large energy transfer from the subresistive velocity eddies to the subresistive magnetic field. This is possible since the fluid energy is formally infinite in (2), and this does not affect the inertial interval. In a situation where the velocity correlation time is not infinitely small, the asymptotic tail $H_k \sim k^{-1-\beta}$ will hold up to the scales where this correlation time is comparable to the resistive time, $k \sim 1/\sqrt{\eta\tau_{\rm corr}}$. On the smaller scales, the magnetic energy spectrum is expected to have a steeper decay, see, e.g., Refs. [7,24,25].

Conclusions.—We propose that magnetic fluctuations are always amplified in isotropic and homogeneous three-dimensional turbulence if the magnetic Reynolds number is large enough. The required critical magnetic Reynolds number sharply increases with velocity roughness, which explains the "no-dynamo" outcomes reported in numerical simulations with small Prandtl numbers. According to our analysis there is no physical reason for the absence of dynamo in numerical simulations other than lack of resolution. Our results also suggest that obtaining small-scale dynamo will be a serious challenge for laboratory experiments, where the magnetic Prandtl number is small, $\mathrm{Pm} \sim 10^{-5}$, and the magnetic Reynolds number is moderate, $\mathrm{Rm} \sim 100$ [26–29].

As we mentioned in the introduction, the Batchelor analogy of magnetic field lines and vorticity lines would hold for $\nu = \eta$ if the initial magnetic field were proportional to the vorticity field. In this special case, magnetic energy would not be amplified. In the model we investigate there are infinitely many such special initial conditions; any function $H_L(r)$ lacking the growing eigenfunctions in its expansion will not be amplified. This, of course, does not mean that dynamo does not exist.

We also note that the Schrödinger-type equation of the form (7) was considered for the magnetic field correlator in the case of a general velocity field in [13]. In this sense the applicability of the equation may not be restricted to the Gaussian, short-time correlated velocity field only. For example, we also investigated the modified Kazantsev model, with a more realistic, finite-time correlated velocity field. The velocity correlation time at a given wave number, k, was chosen to be of the order of the eddy turnover time at the corresponding scale. This model could not be solved analytically; however, its numerical integration gave qualitatively the same results as (1) and (2). We will report these results elsewhere.

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[1] T. G. Cowling, *Magnetohydrodynamics* (Interscience, New York, 1957).
[2] L. D. Landau, E. M. Lifshitz, and L. P. Pitaevskii, *Electrodynamics of Continuous Media* (Elsevier Science, Ltd., Amsterdam, 1985).
[3] R. M. Kulsrud and S. W. Anderson, Astrophys. J. **396**, 606 (1992).
[4] S. I. Vainshtein and F. Cattaneo, Astrophys. J. **393**, 165 (1992).
[5] A. V. Gruzinov and P. H. Diamond, Phys. Rev. Lett. **72**, 1651 (1994).
[6] G. K. Batchelor, Proc. R. Soc. London A **201**, 405 (1950).
[7] R. H. Kraichnan and S. Nagarajan, Phys. Fluids **10**, 859 (1967).
[8] A. P. Kazantsev, Sov. Phys. JETP **26**, 1031 (1968).
[9] A. Schekochihin, S. Boldyrev, and R. M. Kulsrud, Astrophys. J. **567**, 828 (2002); S. Boldyrev, Astrophys. J. **562**, 1081 (2001).
[10] A. Gruzinov, S. Cowley, and R. Sudan, Phys. Rev. Lett. **77**, 4342 (1996).
[11] M. Chertkov, G. Falkovich, I. Kolokolov, and M. Vergassola, Phys. Rev. Lett., **83**, 4065 (1999).
[12] J. Cho, A. Lazarian, and E. T. Vishniac, Astrophys. J. **595**, 812 (2003).
[13] S. I. Vainshtein and L. L. Kichatinov, J. Fluid Mech. **168**, 73 (1986).
[14] A. Tsinober and B. Galanti, Phys. Fluids **15**, 3514 (2003).
[15] K. Ohkitani, Phys. Rev. E **65**, 046304 (2002).
[16] Å. Nordlund *et al.*, Astrophys. J. **392**, 647 (1992).
[17] A. Brandenburg *et al.*, J. Fluid Mech. **306**, 325 (1996).
[18] C. Nore, M. E. Brachet, H. Politano, and A. Pouquet, Phys. Plasmas **4**, 1 (1997).
[19] U. Christensen, P. Olson, and G. A. Glatzmaier, Geophys. J. Int. **138**, 393 (1999).
[20] A. Schekochihin, S. C. Cowley, J. L. Maron, and J. C. McWilliams, astro-ph/0308336.
[21] T. A. Yousef, A. Brandenburg, and G. Rüdiger, astro-ph/0302425.
[22] R. H. Kraichnan, Phys. Fluids **11**, 945 (1968).
[23] L. D. Landau and E. M. Lifshitz, *Quantum Mechanics* (Elsevier Science, Ltd., Amsterdam, 1985).
[24] G. S. Golitsyn, Sov. Phys. Dokl. **5**, 536 (1960).
[25] K. Moffatt, J. Fluid Mech. **11**, 625 (1961).
[26] A. Gailitis *et al.*, Phys. Rev. Lett. **86**, 3024 (2001).
[27] A. Gailitis, O. Lielausis, E. Platacis, G. Gerbeth, and F. Stefani, Rev. Mod. Phys. **74**, 973 (2002).
[28] M. Bourgoin *et al.*, Phys. Fluids **14**, 3046 (2002).
[29] C. B. Forest *et al.*, Magnetohydrodynamics **38**, 107 (2002).