Stabilized Two-Dimensional Vector Solitons

Gaspar D. Montesinos and Víctor M. Pérez-García
Departamento de Matemáticas, Escuela Técnica Superior de Ingenieros Industriales, Universidad de Castilla-La Mancha, 13071 Ciudad Real, Spain

Humberto Michinel
Área de Óptica, Facultade de Ciencias de Ourense, Universidade de Vigo, As Lagoas s/n, Ourense, ES-32005 Spain
(Received 15 October 2003; published 1 April 2004)

In this Letter, we introduce the concept of stabilized vector solitons as nonlinear waves constructed by the addition of mutually incoherent fractions of Townes solitons that are stabilized under the effect of a periodic modulation of the nonlinearity. We analyze the stability of these new kinds of structures and describe their behavior and formation in Manakov-like interactions. Potential applications of our results in Bose-Einstein condensation and nonlinear optics are also discussed.

DOI: 10.1103/PhysRevLett.92.133901 PACS numbers: 42.65.Tg, 03.75.Hh, 05.45.Yv

Since the introduction of the concept of soliton as solitary water waves with robust asymptotic behavior after mutual collisions, many other physical systems have been found with similar dynamics, always described by nonlinear wave equations [1]. For solitons of nonlinear Schrödinger equations (NLSE), the main interest in the early investigations was related to practical applications in optical telecommunications, nowadays well established [2]. The recent interest on solitons in the field of Bose-Einstein condensation (BEC) in alkali gases with negative scattering length [3–6] shows the timeliness of the topic and its central place in modern physics.

Despite the success of the concept of soliton, these structures arise mostly in (1 + 1)-dimensional configurations. In the NLSE case, this is mainly due to the well-known *collapse* property in multidimensional scenarios [7]. In the optical context, *collapse* means that a laser beam with power higher than a critique threshold will be strongly self-focused to a singularity when it propagates in a Kerr-type nonlinear medium, whereas for lower powers it will spread as it propagates. This behavior has also been observed in experiments with matter waves [8].

Since collapse prevents the stability of multidimensional "soliton bullets" in systems ruled by NLSE, great effort has been devoted to search for systems with stable solitary waves in multidimensional configurations [9]. A new way to generate *stabilized* two-dimensional solitary waves has been recently proposed for optical systems [10,11]. The idea is to prevent collapse by using a spatial modulation of the Kerr coefficient (the nonlinearity) of the optical material so that the beam becomes collapsing and expanding in alternating regions and is stabilized in average. The idea has been extended to the field of matter waves in Refs. [12,13]. Finally, in Ref. [14] some general results are provided. Also, in Ref. [15] the NLSE with a time-dependent nonlinearity has been studied.

In the present Letter, we will extend this analysis to the case of mutually incoherent beams with unexpected and surprising results. This is, to our knowledge, the first theoretical evidence of two-dimensional stabilized vector solitons (SVS), a new kind of nonlinear waves which can be constructed in two ways: by direct combination of fractions of Townes solitons or as a result of Manakov interactions [16] between Townes solitons. In both cases, the stabilization against collapse is obtained by the effect of a periodic modulation of the nonlinearity.

The model.—Let us consider an n-component system modeled by equations of the type

$$i\frac{\partial u_j}{\partial t} = -\frac{1}{2}\Delta u_j + g(t)\left(\sum_{k=1}^{n} a_{jk}|u_k|^2\right)u_j, \qquad (1)$$

where $j = 1, \ldots, n$, $u_j{:}\mathbb{R}^+ \times \mathbb{R}^2 \to \mathbb{C}$, $\Delta = \partial^2/\partial x^2 + \partial^2/\partial y^2$, $a_{jk} \in \mathbb{R}$ are the nonlinear coupling coefficients and $g(t)$ is a periodic function accounting for the modulation of the nonlinearity.

Equation (1) is the natural extension of the Manakov system [16] to two transverse dimensions and an arbitrary number of components. In optics, for spatial solitons t is the propagation coordinate and u_j are n mutually incoherent beams. One-dimensional Manakov-type models have been extensively studied in nonlinear optics, mainly due to the potential applications of Manakov solitons in the design of all-optical computing devices [17]. In BEC, these equations (with an additional trapping term) describe the dynamics of multicomponent condensates, u_j being the wave functions for each of the atomic species involved [18,19].

In the scalar case ($n = 1$), it is well known that, if g is constant, there is a stationary radially symmetric solution of Eq. (1) (the so-called Townes soliton): $u(\mathbf{r}, t) = \Phi_\lambda(r)e^{i\lambda t}$. This solution is *unstable* since there are small perturbations of it which lead either to collapse or spreading of the distribution. Because of the scaling invariance of the cubic NLSE, a family of Townes solitons can be generated by making $\Phi_\lambda(r) = \lambda^{1/2}\Phi_1(\lambda^{1/2}r)$.

It is known that an adequate modulation of the nonlinearity stabilizes a Townes soliton yielding to a rapidly oscillating *stabilized* Townes soliton (STS), which we refer hereafter as Φ_S. In this Letter, we use $g(t) = g_0 + g_1 \cos\Omega t$, but we expect that most of our results with similar periodic functions will be qualitatively the same [14].

Stabilized vector solitons.—For a given set of parameters a_{jk} it is possible to use stabilized Townes solitons to build explicit solutions of Eq. (1). These solutions are constructed by taking $u_j = \Phi_{S_j} \equiv \alpha_j \Phi_S \;\; j = 1, \ldots, n$ for any set of coefficients α_j satisfying

$$a_{j1}\alpha_1^2 + \cdots + a_{jn}\alpha_n^2 = 1, \qquad j = 1, \ldots, n. \tag{2}$$

It is not obvious that these *new* solutions will be stabilized by a periodic modulation $g(t)$ of the nonlinearity. Writing $u_j = \Phi_{S_j} + \delta_j$, the equations for δ_j contain cross-modulation terms which could lead to growth of these small perturbations. To test that the stabilization is possible in a wide range of configurations, we have considered several important examples. First, we have studied the most relevant case $n = 2$ and integrated numerically Eq. (1) with different initial data of the form $u_j = \alpha_j \Phi_S$ satisfying (2) and found that these new vector solitons remain stabilized as shown in Fig. 1. From now on, we will name these structures, composed of fractions of Townes solitons, as SVS. We will see below how they emerge in collisions of mutually incoherent STSs, which correspond to the so-called Manakov interactions [16].

We have studied other situations such as a symmetric superposition of four STSs ($n = 4$) with $\alpha_j = 1/\sqrt{4}$ and found similar results. Thus, these structures exist in a wide range of parameters and configurations.

Manakov interactions of STS.—Depending on the mutual velocity of the two interacting STS, we have divided the regime of collisions in two different ones: fast and slow collisions. As we will see below, one of the main results of our work is the possibility of obtaining SVS after *slow* collisions of STS.

First, we have studied collisions of "fast" STSs after which the solitons emerge with only moderate modifications of their amplitude and width as is shown in Fig. 2. It can be seen [Fig. 2(b)] that, during the collision, the soliton becomes spatially asymmetric. An internal asymmetric breathing mode of small amplitude is excited which decays at longer times (not shown in the figure) to the "normal" symmetric breathing mode shown by STSs.

These behaviors can be accounted for by an averaged Lagrangian approach. Equation (1), when $a_{jk} = a_{kj}$, can be obtained from the Lagrangian density

$$\mathcal{L} = \frac{i}{2}\left(u_1 \frac{\partial u_1^*}{\partial t} + u_2 \frac{\partial u_2^*}{\partial t} + \text{H.c.}\right) + \frac{1}{2}|\nabla u_1|^2 + \frac{1}{2}|\nabla u_2|^2 + \frac{g(t)}{2}(a_{11}|u_1|^4 + 2a_{12}|u_1|^2|u_2|^2 + a_{22}|u_2|^4).$$

We choose a simple ansatz accounting for head-on symmetric collisions of equal stabilized solitons moving with

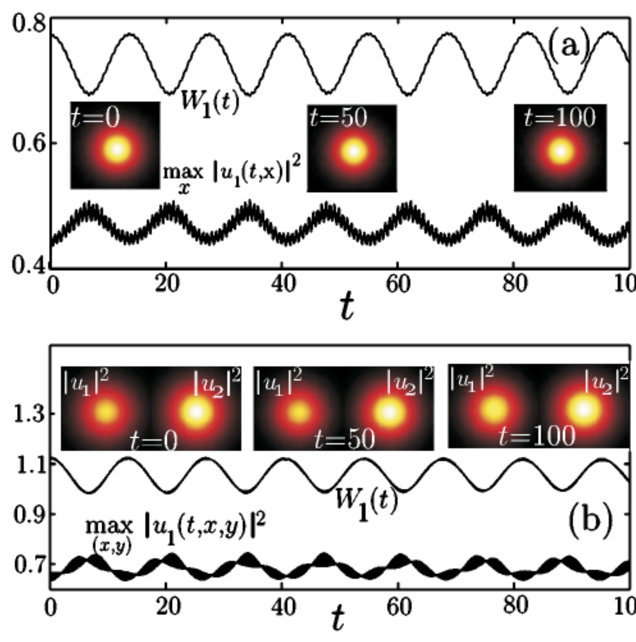

FIG. 1 (color online). (a) Stabilized vector solitons for $\alpha_1 = \alpha_2 = 1/\sqrt{2}$ (the evolution for u_2, not shown here, is very similar). Shown are the time evolution of the width $W_1 = [\int_{\mathbb{R}^2}(x^2 + y^2)|u_1|^2]^{1/2}$ and amplitude $\max_{x\in\mathbb{R}^2}|u_1(x,t)|$. The insets show pseudocolor plots of $|u_1(t,x,y)|^2$ for different times. (b) Same as (a) but for $\alpha_1 = 1/\sqrt{3}$, $\alpha_2 = \sqrt{2/3}$. For both cases $g(t) = -2\pi + 8\pi\cos(40t)$.

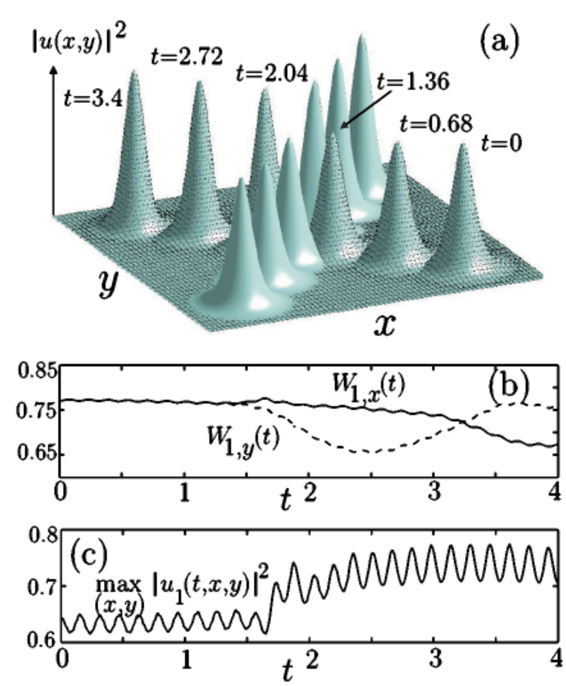

FIG. 2 (color online). Fast collisions of stabilized Townes solitons. Initial data are $u_1(0,\mathbf{r}) = e^{i\mathbf{v}_1\mathbf{r}}\Phi(|\mathbf{r} + \mathbf{r}_1|)$, $u_2(0,\mathbf{r}) = e^{i\mathbf{v}_2\mathbf{r}}\Phi(|\mathbf{r} + \mathbf{r}_2|)$ with $\mathbf{v}_1 = (5/\sqrt{2}, 5/\sqrt{2})$, $\mathbf{v}_2 = (-5/\sqrt{2}, 5/\sqrt{2})$ and $\mathbf{r}_1 = (-6,-6)$, $\mathbf{r}_2 = (6,-6)$. (a) Surface plots of $|u_1|^2$ and $|u_2|^2$ for different times. (b) Evolution of the widths $W_{1,x}(t) = [\int(x - \langle x\rangle)^2|u_1(x,y,t)|^2]^{1/2}$, $W_{1,y}(t) = [\int(y - \langle y\rangle)^2|u_1(x,y,t)|^2]^{1/2}$. (c) Evolution of the maximum amplitude $\max_{(x,y)\in\mathbb{R}^2}|u_1(x,y,t)|$.

opposite speeds and centered on $(-\ell, 0)$ and $(\ell, 0)$:

$$u_1 = Ae^{-(x-\ell)^2/2\omega_x^2 - y^2/2\omega_y^2 + i\beta_x x^2 + i\beta_y y^2} e^{ivx}, \tag{3a}$$

$$u_2 = Ae^{-(x+\ell)^2/2\omega_x^2 - y^2/2\omega_y^2 + i\beta_x x^2 + i\beta_y y^2} e^{-ivx}. \tag{3b}$$

Although Gaussians do not have the right asymptotic decay as STSs, our choice simplifies the calculations and is enough for our present objectives. The standard procedure [20] leads to the equations (for $a_{jk} = 1$)

$$\ddot{\ell} = \ell \frac{Ng(t)}{\pi w_x^3 w_y} e^{-2\ell^2/w_x^2}, \tag{4a}$$

$$\ddot{w}_x = \frac{1}{\omega_x^3} + \frac{Ng(t)}{2\pi\omega_x^2\omega_y}\left[1 + e^{-2\ell^2/w_x^2}\left(1 - \frac{4\ell^2}{\omega_x^2}\right)\right], \tag{4b}$$

$$\ddot{w}_y = \frac{1}{w_y^3} + \frac{Ng(t)}{2\pi w_x w_y^2}(1 + e^{-2\ell^2/w_x^2}), \tag{4c}$$

together with the complementary relations $\beta_j = \dot{w}_j/2\omega_j$, $(j = x, y)$, $v = \dot{\ell} - 2\ell\beta_x$, and the conservation law $N(t) = \pi|A|^2\omega_x\omega_y = \pi|A(0)|^2\omega_x(0)\omega_y(0)$. The different terms in Eqs. (4) account for the phenomenology shown in Fig. 2 and other "fast collisions" studied. For example, they contain the asymmetric interaction [notice the differences between Eqs. (4b) and (4c)] due to the fact that both solitons approach along the x axis and thus become more elongated along that direction as seen in Fig. 2(b). We have numerically integrated Eqs. (4) for fast collisions taking as initial data stabilized Gaussian functions [11,14] and have found results similar to those shown in Fig. 2.

The regime of slow collisions is in the range $|v_2 - v_1| \sim 3$. In this case, the collisions of STSs lead to their splitting into two parts, and formation of two vector solitons takes place as shown in Figs. 3(a) and 3(b). It is remarkable and one of the main results of the paper that the collision mechanism allows the complex coherent rearrangement necessary for the formation of the vector solitons. The fraction of the soliton distributed in the two parts is a function of the only relevant parameter for direct collisions $|v_2 - v_1|$ (due to the Galilean invariance) as shown in Fig. 4(a). In the range $0.2 < |v_2 - v_1| < 3$, we observe formation of two vector solitons which seem to be either unstable or performing high amplitude oscillations for higher speeds and stable in the lower range of speeds (approximately $0.2 < |v_2 - v_1| < 1.2$). If the speed is decreased further, we observe two outgoing vector solitons with complex transient dynamics and nontrivial dependence of the fraction transferred as a function of $|v_2 - v_1|$.

Finally, if the initial speed of the colliding solitons is very small or zero, we have observed a quasibound state of two SVSs which shows several recurrent collisions as shown in Fig. 5. From our simulations, we cannot conclude if this is a true bound state or it finally decays to vector solitons. In Fig. 4(b), we summarize the results of our numerical exploration of STS collisions.

We want to point out that Eqs. (4) provide a reasonably good description of the phenomena described here as far as the ansatz given by Eqs. (3) can describe these complex dynamical behaviors. An example: For very low speeds, the variational equations predict the formation of an oscillating bound state of two STSs. Although this is not the real behavior (a bound state of two SVSs is formed), we get bound states.

The formation of vector solitons from stabilized scalar solitons is a nontrivial phenomenon since there is a delicate balance of both components which must be satisfied in order to avoid destabilization either to collapse or expansion of these structures. It is curious that the system is able to interchange just the right amount of energy to keep both solitons bounded. In fact, the collision mechanism described here can be seen as a way to generate appropriate stabilized vector solitons up from STSs which could be otherwise difficult to obtain.

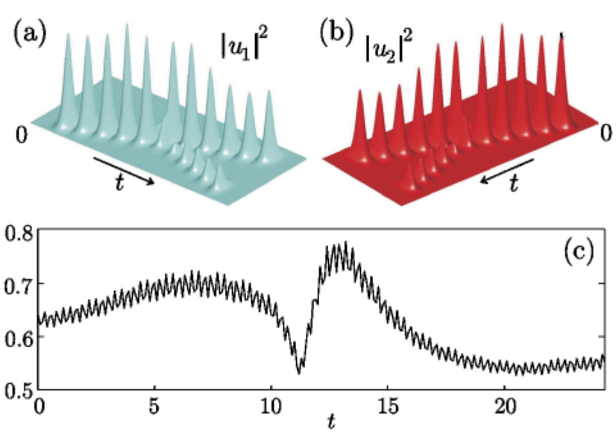

FIG. 3 (color online). Head-on collisions of STSs for initial data $v_1 = -v_2 = 0.3$. (a) Surface plots of $|u_1|^2$. (b) Surface plots of $|u_2|^2$. (c) Evolution of the maximum amplitude $\max_{(x,y)\in\mathbb{R}^2}|u_1(x, y, t)|$.

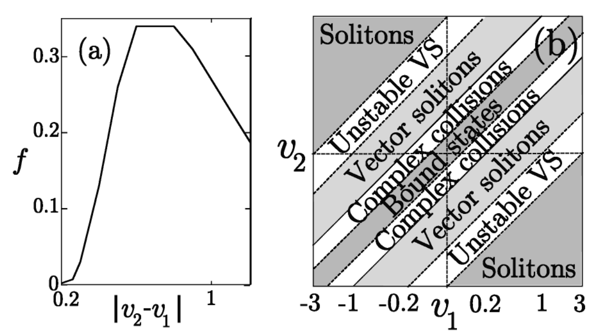

FIG. 4. Asymptotic behavior after head-on collisions of STSs of the form $u_1 = \Phi(|\mathbf{r} - \ell|)e^{iv_1x}$, $u_2 = \Phi(|\mathbf{r} + \ell|)e^{iv_2x}$ and large enough ℓ ($\sim$4). (a) Regimes of behavior as functions of v_1, v_2. (b) Quotient (f) of the squared amplitudes of the small and the large peaks which are generated after the collision when a VS is formed [see Fig. 3] for the regime of speeds in the range $0.2 \leq |v_2 - v_1| \leq 1.2$.

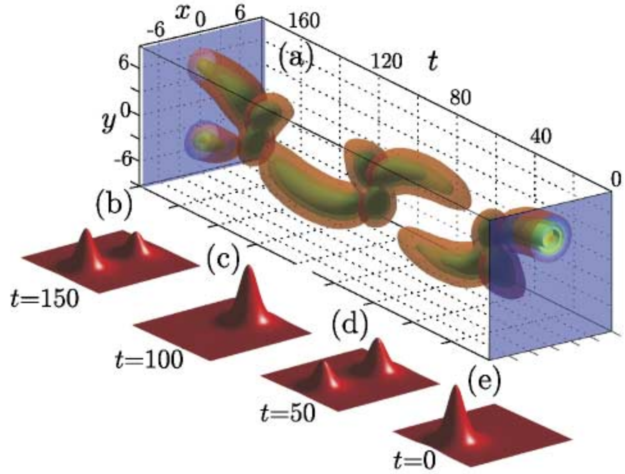

FIG. 5 (color online). Oscillations after a collision of STSs with $v_1 = v_2 = 0$. (a) Isosurface plot of $|u_1(t, x, y)|^2$ (shown are isosurfaces corresponding to 0.05, 0.15, and 0.25). (b)–(e) Surface plots of $|u_1(t, x, y)|^2$ for (b) $t = 0$, (c) $t = 50$, (d) $t = 100$, and (e) $t = 150$. The corresponding evolution for u_2 is symmetric with respect to the y axis.

It is also remarkable that no collapse phenomena are observed in our simulations, rather instead most of the collisions observed lead to remarkably robust scalar or vector solitons. This is very different from what happens in coherent collisions of STSs which lead to collapse.

Applications to optical systems.—In optics, a simple way to obtain a periodically varying nonlinearity is to use a collection of parallel planar plates, corresponding to two different nonlinear optical materials. The functions u_1 and u_2 can be seen as two orthogonal states of polarization (which in the absence of nonlinear birefringence leads to symmetric a_{ij}). Taking our initial width to correspond to a laser spot of 100 μm, the plates should be about 2.5 μm thick in order to reproduce the $g(t)$ modulation. For a typical Nd:YAG with $\lambda = 1.064$ μm and 1 GW peak power, adequate materials for the plates would be GaAs ($n_2 = -3 \times 10^{-13}$ cm^2/W) and 4-dimethylamino-4-nitrostilbene polymer [21] ($n_2 = 2 \times 10^{-13}$ cm^2/W). Using this system, one should be able to observe the striking propagation of a stable vector soliton in Kerr media.

We must stress that the limit $n \to \infty$ of our model could be used to study nonlinear propagation of totally incoherent light.

This work has been partially supported by Ministerio de Ciencia y Tecnología (MCyT) under Grants No. BFM2000-0521, No. BFM2003-02832, and No. TIC-2000-1105-C03-01, and Consejería de Ciencia y Tecnología de la Junta de Comunidades de Castilla-La Mancha under Grant No. PAC-02-002. G. M. is supported by Grant No. AP2001-0535 from MECD.

[1] See N. N. Akhmediev and A. Ankiewicz, *Solitons: Nonlinear Pulses and Beams* (Chapman and Hall, London, 1997); A.W. Snyder and D. J. Mitchell, Science **276**, 1538 (1997); G. I. Stegeman and M. Segev, Science **286**, 1518 (1999).
[2] Yu. S. Kivshar and G. P. Agrawal, *Optical Solitons: From Fibers to Photonic Crystals* (Academic, San Diego, 2003); *Optical Solitons: Theoretical Challenges and Industrial Perspectives*, edited by V. E. Zakharov and S. Wabnitz (Springer-Verlag, Berlin, 1999).
[3] V. M. Pérez-García, H. Michinel, and H. Herrero, Phys. Rev. A **57**, 3837 (1998).
[4] K. E. Strecker, G. B. Partridge, A. G. Truscott, and R. G. Hulet, Nature (London) **417**, 150 (2002).
[5] L. Khaykovich, F. Schreck, G. Ferrari, T. Bourdel, J. Cubizolles, L. D. Carr, Y. Castin, and C. Salomon, Science **296**, 1290 (2002).
[6] L. Salasnich, A. Parola, and L. Reatto, Phys. Rev. Lett. **91**, 080405 (2003).
[7] C. Sulem and P. Sulem, *The Nonlinear Schrödinger Equation: Self-Focusing and Wave Collapse* (Springer-Verlag, Berlin, 2000).
[8] C. C. Bradley, C. A. Sackett, J. J. Tollett, and R. G. Hulet, Phys. Rev. Lett. **75**, 1687 (1995); E. A. Donley, N. R. Claussen, S. L. Cornish, J. L. Roberts, E. A. Cornell, and C. E. Wieman, Nature (London) **412**, 295 (2001).
[9] F. Wise and P. Di Trapani, Opt. Photon. News **13**, 28 (2002); M. Segev and G. Stegeman, Phys. Today **51**, No. 8, 42 (1998).
[10] L. Berge, V. K. Mezentsev, J. J. Rasmussen, P. L. Christiansen, and Y. B. Gaididei, Opt. Lett. **25**, 1037 (2000).
[11] I. Towers and B. A. Malomed, J. Opt. Soc. Am. B **19**, 537 (2002).
[12] H. Saito and M. Ueda, Phys. Rev. Lett. **90**, 040403 (2003).
[13] F. Kh. Abdullaev, J. G. Caputo, R. A. Kraenkel, and B. A. Malomed, Phys. Rev. A **67**, 013605 (2003).
[14] G. D. Montesinos, V. M. Pérez-García, and P. Torres, arxiv.org/nlin.PS/0305030 [Physica D (Amsterdam) (to be published)].
[15] F. Kh. Abdullaev and R. A. Kraenkel, Phys. Lett. A **272**, 395 (2000).
[16] S.V. Manakov, Zh. Eksp. Teor. Fiz. **65**, 505 (1973) Sov. Phys. JETP **38**, 248 (1974).
[17] See, e.g., T. Kanna and M. Lakshmanan, Phys. Rev. E **67**, 046617 (2003); N. Akhmediev, W. Krolikowski, and A.W. Snyder, Phys. Rev. Lett. **81**, 4632 (1998); Y. Tan and J. Yang, Phys. Rev. E **67**, 056616 (2001).
[18] D. S. Hall, M. R. Matthews, J. R. Ensher, C. E. Wieman, and E. A. Cornell, Phys. Rev. Lett. **81**, 1539 (1998).
[19] B. D. Esry and C. H. Greene, Nature (London) **392**, 434 (1998).
[20] D. Anderson and M. Lisak, Phys. Rev. A **32**, 2270 (1985); B. A. Malomed, Prog. Opt. **43**, 71 (2002).
[21] E.V. Tomme, P. P.V. Daele, R. G. Baets, and P. E. Lagasse, IEEE J. Quantum Electron. **27**, 778 (1991).