

Determining a Quantum State by Means of a Single Apparatus

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(Received 20 November 2003; published 25 March 2004)

The unknown state $\hat{\rho}$ of a quantum system S is determined by letting it interact with an auxiliary system A , the initial state of which is known. A one-to-one mapping can thus be realized between the density matrix $\hat{\rho}$ and the probabilities of the occurrence of the eigenvalues of a single and factorized observable of $S + A$, so that $\hat{\rho}$ can be determined by repeated measurements using a single apparatus. If S and A are spins, it suffices to measure simultaneously their z components after a controlled interaction. The most robust setups are determined in this case for an initially pure or a completely disordered state of A . They involve an Ising or anisotropic Heisenberg coupling and an external field.

DOI: 10.1103/PhysRevLett.92.120402

PACS numbers: 05.30.-d, 05.70.Ln

Consider a set of identical quantum systems S , prepared in some unknown state $\hat{\rho}$. Measuring some observable $\hat{\omega}$ of S provides the probabilities $p_i = \text{tr} \hat{\rho} \hat{\pi}_i$ for the distinct eigenvalues ω_i of $\hat{\omega}$, where $\hat{\pi}_i$ are the associated eigenprojections. We thus find partial information on $\hat{\rho}$. A question then arises which lies at the heart of quantum theory: Which observables need to be measured for a complete determination of $\hat{\rho}$ [1–4]? The recognition that certain noncommuting observables have to be measured for that purpose was used by Bohr to formulate the principle of complementarity [1,5]. Later on the problem of determining an unknown state was considered from various perspectives for continuous [3] and discrete systems [4] and found applications in quantum communication [6,7]. But whether noncommutative measurements are truly needed was rarely questioned.

We wish to determine the whole set of unknown matrix elements of the state $\hat{\rho}$ of a system S . In fact, what we call “the state of a system,” whether it is pure or not, refers to the density matrix describing an ensemble, and the determination of the data will require repeated experiments. However, we will show that a single apparatus which measures commuting observables only suffices for our purposes. The key of the method consists of coupling the system S to an auxiliary system A , the state of which we know. We will show that the full $\hat{\rho}$ can then be deduced from simultaneous measurements of two obviously commuting observables, $\hat{\omega}$ pertaining to S and \hat{o} pertaining to A , respectively. Repeated measurements of $\hat{\omega}$ and \hat{o} yield the joint probabilities of occurrence for all pairs of eigenvalues of $\hat{\omega}$ and \hat{o} ; this will be sufficient to determine the whole density matrix $\hat{\rho}$ of S , provided the initial state of A and the joint evolution of $S + A$ are known. We shall discuss the most robust measurements of this type and point out certain advantages that this scheme provides over the usual methods of state determination. Realistic experiments can be designed along these lines, as we shall see.

General reasoning.—The following counting argument already suggests the feasibility of the above idea: The number of real parameters that characterize $\hat{\rho}$, a Hermitian matrix with unit trace in m -dimensional Hilbert space, is $m^2 - 1$. On the other hand, the most informative repeated measurements of an observable $\hat{\omega}$ of S , those for which the spectrum ω_i is nondegenerate, provide $m - 1$ independent data, the probabilities $\text{tr} \hat{\rho} \hat{\pi}_i$. Thus, measurements directly performed on S must deal with at least $(m^2 - 1)/(m - 1) = m + 1$ noncommuting observables to fully determine the unknown $\hat{\rho}$. For instance, the 2×2 density matrix $\hat{\rho} = \frac{1}{2}(\hat{1} + \vec{\rho} \cdot \vec{\hat{\sigma}})$ of a spin- $\frac{1}{2}$ is parametrized by the $m^2 - 1 = 3$ expectation values $\vec{\rho} = \text{tr}(\hat{\rho} \cdot \vec{\hat{\sigma}})$ of the Pauli matrices $\vec{\hat{\sigma}} = (\hat{\sigma}_1, \hat{\sigma}_2, \hat{\sigma}_3)$ with $\hat{\sigma}_1 \hat{\sigma}_2 = i \hat{\sigma}_3$, and its determination requires measuring the spin in $m + 1 = 3$ noncoplanar directions. Experiments with three different apparatuses are needed, even if the unknown state $\hat{\rho}$ is pure ($\hat{\rho}^2 = \hat{\rho}$ or $\vec{\rho}^2 = 1$), since then the measurement of two components of $\vec{\rho}$ does not fix the sign of the third one.

In order to design a scheme where $\hat{\rho}$ will be determined by measurements of a single observable (or, equivalently, by commuting measurements only), it is natural to introduce [8] an auxiliary system A that we term the assistant and which lies in a known state \hat{r} . Let n be the number of dimensions of the Hilbert space of A . The compound system $S + A$ has a $mn \times mn$ density matrix $\hat{\mathcal{R}} = \hat{\rho} \otimes \hat{r}$. Reference [8] proposes measuring one of its observables, a “universal quantum observable,” $\hat{\Omega} = \sum \Omega_\alpha \hat{P}_\alpha$, where the spectrum Ω_α is not degenerate so that the eigenprojections \hat{P}_α constitute a complete set of $mn - 1$ commuting observables. Such repeated measurements provide $mn - 1$ independent data $P_\alpha = \text{tr} \hat{\mathcal{R}} \hat{P}_\alpha$, the probabilities of the eigenvalues Ω_α of $\hat{\Omega}$. A linear mapping $\hat{\rho} \mapsto P_\alpha = \sum_{ij} M_{\alpha,ij} \rho_{ji}$ is thus generated, leading from the $m^2 - 1$ real parameters of $\hat{\rho}$ to the $mn - 1$ data P_α . If the assistant A has the same dimensionality m as S , this mapping is represented by a square matrix. In

general, if the measured observable $\hat{\Omega}$ intertwines sufficiently S and A , the determinant $\Delta = \det M_{\alpha,ij}$ of this matrix will be nonzero. The inverse mapping then solves our problem: measurements of $\hat{\Omega}$, performed repeatedly with a single apparatus, yield the probabilities P_α , the knowledge of which is equivalent to that of $\hat{\rho}$.

However, the above scheme is very difficult to implement in practice, since it implies measuring an observable $\hat{\Omega}$ (or a commuting set \hat{P}_α) which thoroughly mixes S and A . We propose here a procedure allowing a much simpler choice for $\hat{\Omega}$. The measurement of $\hat{\Omega}$ is performed not at the time $t = 0$, at which S is prepared in the unknown state $\hat{\rho}$ and A in the known state \hat{r} , but at a later time $t = \tau$. During the lapse $0 < t < \tau$, S and A interact, their evolution being generated by a known Hamiltonian \hat{H} . The state of the composite system $S + A$ which is tested is now $\hat{R}_\tau = \hat{U}\hat{R}_0\hat{U}^\dagger$, where the initial state is $\hat{R}_0 = \hat{\rho} \otimes \hat{r}$ and the evolution operator is $\hat{U} = e^{-i\hat{H}\tau}$. The required mixing of $\hat{\rho}$ and \hat{r} being thus achieved by dynamics, we can now measure the simplest possible nondegenerate observable $\hat{\Omega}$, a factorized quantity $\hat{\Omega} = \hat{\omega} \otimes \hat{\delta}$. The observables $\hat{\omega}$ and $\hat{\delta}$ of S and A have the spectral decompositions $\hat{\omega} = \sum_{i=1}^m \omega_i \hat{\pi}_i$ and $\hat{\delta} = \sum_{a=1}^n o_a \hat{p}_a$ and the projection operator \hat{P}_α , with $\alpha = \{ia\}$, takes the form $\hat{P}_\alpha \equiv \hat{P}_{ia} = \hat{\pi}_i \otimes \hat{p}_a$. Repeated measurements of $\hat{\Omega}$, i.e., of $\hat{\omega}$ and $\hat{\delta}$ simultaneously, determine the joint probabilities

$$P_\alpha \equiv P_{ia} = \text{tr } \hat{U}(\hat{\rho} \otimes \hat{r})\hat{U}^\dagger(\hat{\pi}_i \otimes \hat{p}_a) \quad (1)$$

to observe ω_i for S and o_a for A . [The numbers P_{ia} are the diagonal elements of $\hat{U}(\hat{\rho} \otimes \hat{r})\hat{U}^\dagger$ in the factorized basis which diagonalizes $\hat{\omega}$ and $\hat{\delta}$.] Like above, the mapping $\hat{\rho} \mapsto P_\alpha$ is expected to be invertible for $n \geq m$, provided \hat{H} couples S and A sufficiently. We shall see that even very simple interactions can achieve the thorough mixing that is required. Then simultaneous measurements of $\hat{\omega}$ on the system S and of $\hat{\delta}$ on the assistant A , based on the mere counting of the events $\{ia\}$ and of their correlations, fully determine $\hat{\rho}$ through inversion of the equation (1).

For given observables $\hat{\omega}$ and $\hat{\delta}$ and for a given initial state \hat{r} of the assistant, the precision of this scheme of measurement of $\hat{\rho}$ relies on the ratio between the experimental uncertainty about the set P_α and the resulting uncertainty on $\hat{\rho}$, which can be characterized by the determinant Δ of the transformation (1). For $\Delta = 0$ it would be impossible to determine $\hat{\rho}$ by means of P_α . The Hamiltonian \hat{H} and the duration τ of the interaction should thus be chosen so as to maximize $|\Delta|$.

Two by two density matrix.—We illustrate the above ideas by studying a two-level system S ($m = 2$). We use the spin- $\frac{1}{2}$ representation $\hat{\rho} = \frac{1}{2}(\hat{1} + \vec{\rho} \cdot \vec{\hat{\sigma}})$. The determination of the unknown polarization vector $\vec{\rho}$ relies on the coupling of S with the assistant A , which we first take as another two-level system ($n = 2$). The observables $\hat{\omega}$ and $\hat{\delta}$ to be measured are the z components $\hat{\sigma}_3$ and \hat{s}_3 of S and A , which may be equal to 1 or -1 . The projection operators are $\hat{\pi}_i = \frac{1}{2}(\hat{1} \pm \hat{\sigma}_3)$ and $\hat{p}_a = \frac{1}{2}(\hat{1} \pm \hat{s}_3)$ for i and a

equal to ± 1 . Experiments determine the four joint probabilities $P_\alpha = \{P_{++}, P_{+-}, P_{-+}, P_{--}\}$, for σ_3 and s_3 to equal 1 or -1 . These probabilities are related to the three real parameters $\vec{\rho}$ of $\hat{\rho}$ through Eq. (1), which reads

$$P_\alpha = u_\alpha + \vec{v}_\alpha \cdot \vec{\rho}, \quad (2)$$

$$u_\alpha = \frac{1}{2}[\hat{U}(\hat{1} \otimes \hat{r})\hat{U}^\dagger]_{\alpha,\alpha}, \quad \vec{v}_\alpha = \frac{1}{2}[\hat{U}(\hat{\sigma} \otimes \hat{r})\hat{U}^\dagger]_{\alpha,\alpha},$$

with $\alpha = \{ia\} = \{\pm\pm\}$ and matrix elements taken in the standard representation of the Pauli matrices $\hat{\sigma}$ and \hat{s} .

By construction, the mapping (2) is such the probabilities P_α are non-negative and normalized for any $\hat{\rho}$ such that $\vec{\rho}^2 \leq 1$. These properties are expressed by

$$u_\alpha \geq |\vec{v}_\alpha|, \quad \sum_\alpha u_\alpha = 1, \quad \sum_\alpha \vec{v}_\alpha = 0. \quad (3)$$

The determinant Δ of the transformation $\hat{\rho} \mapsto P_\alpha$ is 4 times the volume of the parallelepiped having any three of the four vectors \vec{v}_α as its sides, e.g., $\Delta = 4\vec{v}_{++} \cdot (\vec{v}_{-+} \times \vec{v}_{+-})$. Provided the evolution operator \hat{U} is such that the vectors \vec{v}_α are not coplanar, the transformation (2) can be inverted, and $\hat{\rho}$ is deduced from the set P_α of classical probabilities. Alternatively, $\hat{\rho}$ is deduced from the expectation values $\langle \hat{\sigma}_3 \rangle$, $\langle \hat{s}_3 \rangle$, and $\langle \hat{\sigma}_3 \hat{s}_3 \rangle$ at the time $t = \tau$, which are simultaneously measurable and are in one-to-one correspondence with the set P_α .

We first look for the upper bound of $|\Delta|$ implied by the conditions (3). First we note that $|\Delta|$ increases with $|\vec{v}_\alpha|$ for each α . We therefore maximize Δ^2 under the constraints $\sum_\alpha |\vec{v}_\alpha| = 1$ and $\sum_\alpha \vec{v}_\alpha = 0$ that we account for by means of Lagrange multipliers ν and $\vec{\mu}$. Varying $\frac{1}{8}\Delta^2 + \lambda \sum_\alpha |\vec{v}_\alpha| + \vec{\mu} \cdot \sum_\alpha \vec{v}_\alpha$ and eliminating $\vec{\mu}$, we find

$$\Delta(\vec{v}_{+-} \times \vec{v}_{-+}) = \nu \left(\frac{\vec{v}_{++}}{|\vec{v}_{++}|} - \frac{\vec{v}_{--}}{|\vec{v}_{--}|} \right), \quad (4)$$

and other equations resulting from all permutations of the α 's. This yields symmetric solutions for which the four vectors \vec{v}_α , $\alpha = \{\pm, \pm\}$ form a regular tetrahedron:

$$u_\alpha = |\vec{v}_\alpha| = \frac{1}{4}, \quad \frac{\vec{v}_\alpha \cdot \vec{v}_\beta}{|\vec{v}_\alpha||\vec{v}_\beta|} = -\frac{1}{3}, \quad \alpha \neq \beta. \quad (5)$$

These solutions are not unique: they follow from one another by rotation in the space of the spins $\vec{\sigma}$ and permutation of the indices α . The corresponding upper bound for the determinant is $|\Delta| = 1/(12\sqrt{3})$.

Let us exhibit a measurement scheme that allows one to reach this bound. We have to construct a unitary operator \hat{U} that satisfies Eqs. (2) and (5), and to find a Hamiltonian \hat{H} and an interaction time τ such that $\hat{U} = e^{-i\hat{H}\tau}$. We assume that the assistant is initially in the pure state $\hat{r} = \hat{p}_+ = \frac{1}{2}(1 + \hat{s}_3)$, and we orient the tetrahedron \vec{v}_α in the direction $\vec{v}_{+\pm} = (\pm 1, 1, \pm 1)/4\sqrt{3}$, $\vec{v}_{-\pm} = (\pm 1, -1, \mp 1)/4\sqrt{3}$. The correspondence (2) then takes an especially simple form:

$$\rho_1 = \sqrt{3}\langle \hat{s}_3 \rangle, \quad \rho_2 = \sqrt{3}\langle \hat{\sigma}_3 \rangle, \quad \rho_3 = \sqrt{3}\langle \hat{\sigma}_3 \hat{s}_3 \rangle, \quad (6)$$

yielding directly the density matrix $\hat{\rho}$ in terms of the expectation values and the correlation of the commuting observables $\hat{\sigma}_3$ and \hat{s}_3 in the final state. It is easy to verify that this correspondence can be achieved under the action of the Hamiltonian $\hat{H} = \hat{\sigma}_1(\hat{s}_1 \cos\phi + \hat{s}_3 \sin\phi)/\sqrt{2} + \frac{1}{2}[(\hat{s}_2 - \hat{s}_1)\sin\phi + \hat{s}_3 \cos\phi]$, where $2\phi = 0.95531$ is the angle between \vec{v}_{++} and the z axis, that is, $\cos 2\phi = 1/\sqrt{3}$. Noting that $\hat{H}^2 = \sin^2\chi$, where $\chi = 1.11069$ satisfies $\cos\chi = \frac{1}{2}\cos\phi$, and taking as duration of the evolution $\tau = \chi/\sin\chi$, we obtain $\hat{U} \equiv \exp(-i\hat{H}\tau) = \cos\chi - i\hat{H}$. Insertion in (2) allows us to check Eq. (5) and to get the expected optimal correspondence (6). The simpler form,

$$\hat{H} = \frac{1}{\sqrt{2}}\hat{\sigma}_1\hat{s}_1 + \frac{1}{2}(\hat{s}_2 \sin\phi + \hat{s}_3), \quad (7)$$

follows by a rotation of $\vec{\hat{s}}$ and also achieves an optimal mapping $\hat{\rho} \mapsto P_\alpha$, provided $\hat{s}_3 \rightarrow \hat{s}_1 \sin\phi + \hat{s}_3 \cos\phi$ both in the measured projections $\hat{p}_\alpha = \frac{1}{2}(1 \pm \hat{s}_3)$ and in the initial state $\hat{r} = \hat{p}_+$. The first term in (7) describes, in the spin language, an Ising coupling, while the second term represents a transverse magnetic field acting on A .

Larger assistant.—We have optimized above the determination of $\hat{\rho}$ by coupling S to an assistant A that starts in a pure state and has the same dimension $n = m = 2$ as S . Let us see how the quality of the measurement, as expressed by the magnitude of Δ , depends on these conditions. Consider, for instance, an assistant consisting of q spins, in which case $m = 2$ and $n = 2^q > m$. We now denote as \hat{s}_3 some two-valued observable of A which is subjected to measurement at the time τ . The only changes in (1) are the dimension n of the matrix \hat{r} and the fact that the two projection operators \hat{p}_α no longer constitute a complete set in the Hilbert space of A . Experiment still provides the four probabilities $P_\alpha = \text{tr } \hat{\mathcal{R}}_\tau(\hat{\pi}_i \otimes \hat{p}_\alpha)$ with unit sum, and the mapping $\hat{\rho} \mapsto P_\alpha$ keeps the form (2). The conditions (3) still hold, since they express simply that the correspondence (1) or (2) preserves the positivity and the normalization. When obtaining the upper bound $1/(12\sqrt{3})$ for $|\Delta|$ we relied only on these conditions. Therefore, using a larger assistant cannot improve upon the optimal solutions found for $n = 2$ and pure \hat{r} . In all cases, the maximum of $|\Delta|$ is reached for mappings (2) which involve the regular tetrahedron (5). In such mappings there exist four pure states, $\hat{\rho}_{\vec{\alpha}} = \frac{1}{2}(1 - 4\vec{v}_{\vec{\alpha}} \cdot \vec{\hat{\sigma}})$, for which one probability, $P_{\vec{\alpha}}$, vanishes.

If the measured observable \hat{o} of A has more than two distinct eigenvalues, the probabilities P_{ia} , which depend only on the three parameters of $\hat{\rho}$, are related to one another. This opens a possibility of improving the determination of $\hat{\rho}$ through a cross-check of the data P_{ia} .

Assistant in a mixed state.—Returning to the case $n = m = 2$ we now look how much $|\Delta|$ decreases when the state $\hat{r} = \frac{1}{2}(\hat{1} + \lambda\hat{s}_3)$ is no longer pure ($0 \leq \lambda < 1$). We find it convenient to parametrize \hat{U} in terms of three angles, θ , φ , and ψ , and four unit vectors $\vec{\chi}$, $\vec{\xi}$, $\vec{\eta}$, and $\vec{\zeta}$,

$$\begin{aligned} \hat{U} = & (\hat{V}\hat{p}_+ + \hat{W}\hat{p}_-)[\cos\varphi(\cos\theta\hat{s}_3 + \sin\theta\hat{s}_1) \\ & + \sin\varphi(\sin\theta\hat{s}_3 - \cos\theta\hat{s}_1)\vec{\chi} \cdot \vec{\hat{\sigma}}] \\ & \times [\cos\psi + i\sin\psi(\vec{\xi} \cdot \vec{\hat{\sigma}})\hat{s}_3], \end{aligned} \quad (8)$$

where the 2×2 unitary matrices \hat{V} and \hat{W} in the space of S are characterized by $\hat{V}^\dagger\hat{\sigma}_3\hat{V} = \vec{\eta} \cdot \vec{\hat{\sigma}}$, $\hat{W}^\dagger\hat{\sigma}_3\hat{W} = \vec{\zeta} \cdot \vec{\hat{\sigma}}$. The expression of $|\Delta|$ resulting from (2) is a function of these parameters. We maximize it, first with respect to $\vec{\chi}$, $\vec{\xi}$, $\vec{\eta}$, and $\vec{\zeta}$, then to ψ and finally to θ and φ . The calculation is straightforward but tedious and we present only the result for $\vec{\chi} \cdot \vec{\xi} = 0$, which is optimal for λ close to 1 and at $\lambda = 0$:

$$\begin{aligned} (32\Delta)^2 = & 1 - \frac{2\lambda^4}{3} + \frac{8\lambda^6}{9} - \frac{\lambda^8}{27} \\ & + \sqrt{1 + \lambda^2} \left(\frac{2\lambda}{3} \sqrt{3 - \lambda^2} \right)^3. \end{aligned} \quad (9)$$

This is an increasing function of λ , behaving as $\lambda/12\sqrt{3}$ near $\lambda = 1$ and reproducing there the maximum $1/12\sqrt{3}$.

Let us now focus on the extreme situation $\lambda = 0$, for which the state $\hat{r} = \frac{1}{2}\hat{1}$ is completely disordered. It is advantageous to use such a state, as it is easier to prepare than a pure state: one lets the assistant interact with a hot thermal bath; for a spin, one leaves it unpolarized. It would be completely ineffective to use in this case the simple evolution (7) which was optimal for $\lambda = 1$, since any $\hat{\rho}$ would then map onto the trivial set of probabilities $P_\alpha = 1/4$. However, according to (9), $|\Delta|$ may reach the value $1/32$ for a suitable choice of the evolution \hat{U} . This goal is achieved, in particular, for $\theta = \varphi = \pi/8$, $\psi = \pi/4$, $\vec{\eta} = \vec{\zeta} = (\vec{\xi} + \vec{\chi} \times \vec{\xi})/\sqrt{2}$. Choosing $\vec{\chi} = (1, 0, 0)$ and $\vec{\xi} = (0, 1, 0)$, we then find from Eq. (2) the same mapping as (6), except for replacing $\sqrt{3}$ by 2.

Surprisingly, the efficiency of the new scheme is not much worse than when the assistant starts in a pure state. For $\lambda = 0$ the Hamiltonians needed to maximize $|\Delta|$ require a coupling more complicated than (7), e.g.,

$$\hat{H} = \frac{1}{2}\vec{\hat{\sigma}} \cdot \vec{\hat{s}} - \sqrt{2}\hat{\sigma}_2\hat{s}_2 + \frac{1}{\sqrt{2}}(\hat{\sigma}_1 + \hat{s}_1). \quad (10)$$

This Hamiltonian involves an anisotropic Heisenberg interaction and an external field acting symmetrically on S and A . Using Eq. (2) with $\hat{r} = \frac{1}{2}\hat{1}$, one can check that for $\tau = \frac{1}{4}(2k+1)\pi$ the evolution operator $\hat{U} = e^{-i\hat{H}\tau}$ leads to an optimal solution with $\Delta = 1/32$. For $\tau = \pi/4$ Eq. (6) is replaced by

$$\begin{aligned} \rho_1 = & 2\langle\hat{\sigma}_3\hat{s}_3\rangle, & \rho_2 = & 2(\langle\hat{s}_3\rangle\cos\gamma + \langle\hat{\sigma}_3\rangle\sin\gamma), \\ \rho_3 = & 2(\langle\hat{\sigma}_3\rangle\cos\gamma - \langle\hat{s}_3\rangle\sin\gamma), & \gamma = & \frac{\pi(1 + \sqrt{2})}{4}. \end{aligned} \quad (11)$$

Conclusion.—The noncommutative information contained in the density matrix of a quantum system S can be transformed by a one-to-one correspondence into ordinary information associated with a set P_α of ordinary probabilities for exclusive events. The price to be paid is the introduction of an assistant system A . The

correspondence can be experimentally implemented by letting S and A suitably interact, then by performing simultaneous measurements of two commuting observables $\hat{\omega}$ and $\hat{\delta}$ pertaining to S and A , respectively. Counting of events in repeated experiments yields the probabilities P_{ia} for $\hat{\omega}$ to take the value ω_i and for $\hat{\delta}$ to take the value δ_a . Provided the number of distinct eigenvalues ω_i ($1 \leq i \leq m$) of $\hat{\omega}$ equals the dimension m of the Hilbert space of S , and provided the number of eigenvalues of $\hat{\delta}$ is also m (at least), the correspondence $\hat{\rho} \mapsto P_\alpha$ can be inverted, as we displayed on several examples. This condition implies that the assistant has a dimension $n \geq m$. Hence an initially unknown $\hat{\rho}$ can be determined via P_α by means of a single apparatus.

As compared to the standard determination schemes of $\hat{\rho}$ based on direct noncommutative measurements of S , the present method has several advantages. (i) It is more economical, since it involves only one observable $\hat{\omega}$ of S and one observable $\hat{\delta}$ of A , whereas direct determinations require measuring at least $m + 1$ noncommuting observables of S . (ii) This full set of $m + 1$ observables is not always accessible in practice. For instance, for a two-level atom prepared in some unknown state $\hat{\rho}$, ρ_3 is readily measured through the occupation probability of the excited state, but ρ_1 and ρ_2 can be determined only indirectly. Interaction of S with another two-level atom A having a known initial density operator (whose preparation may be straightforward, as we saw above) can provide the full $\hat{\rho}$ through mere simultaneous measurements of the occupation probabilities for the levels of S and A . (iii) Standard statistical and information theoretical methods for dealing with incomplete or noisy experimental data cannot be directly extended to quantum mechanics [9] when results are produced by means of different apparatuses, since the data then pertain to different contexts. The present scheme circumvents this difficulty.

Taking as a criterion of quality of our measurement schemes the size of the determinant Δ of the mapping $\hat{\rho} \mapsto P_\alpha$, we have explored for $m = 2$ the conditions that lead to the best determination of $\hat{\rho}$ for some uncertainty on the set P_α . For an assistant with dimension $n = 2$, its known initial state $\hat{\rho}$ should be pure and the parameters of the Hamiltonian should be suitably chosen as in the example (7). Not much is lost if $\hat{\rho}$ is mixed: Even for a completely disordered assistant with $n = 2$, we can find Hamiltonians such as (10) for which $|\Delta|$ is smaller than the upper bound only by a factor $(\sqrt{3}/2)^3 \simeq 0.65$. The determinant cannot be enlarged by use of an assistant with dimension larger than 2, but then the determination of $\hat{\rho}$ can benefit from measuring redundant data.

For $m = 2$, the optimal mappings (2) and (5) amount to identify, via a dynamical process, the joint probabilities P_α for $\hat{\sigma}_3$ and \hat{s}_3 with the expectation values in the state $\hat{\rho}$ of the observables $\frac{1}{2}(\hat{1} - \hat{\Omega}_\alpha)$, where $\hat{\Omega}_\alpha = \frac{1}{2}(\hat{1} - 4\vec{v}_\alpha \cdot \hat{\vec{\sigma}})$ pertains to S . The four observables $\hat{\Omega}_\alpha$ are projection operators, satisfying $\text{tr} \hat{\Omega}_\alpha \hat{\Omega}_\beta = \frac{1}{3}$ for $\alpha \neq \beta$, and spanning the space of observables $\hat{\omega}$ of S . For $m > 2$ we

conjecture that a bound on the determinant Δ may be found by considering in the Hilbert space of S a set of m^2 projections $\hat{\Omega}_\alpha$, satisfying $\text{tr} \hat{\Omega}_\alpha = 1$, $\text{tr} \hat{\Omega}_\alpha \hat{\Omega}_\beta = 1/(m + 1)$ for $\alpha \neq \beta$, $\sum_\alpha \hat{\Omega}_\alpha = m \hat{1}$, and constituting a basis for the observables $\hat{\omega}$. Then the mapping matrix M in $P_\alpha = \sum_{ij} M_{\alpha,ij} \rho_{ji}$ is expected to be given by $m(m - 1)M_{\alpha,ij} = \delta_{ij} - \Omega_{\alpha,ij}$. This form makes Δ stationary under the constraints imposed by positivity and normalization alone. As above, there are m^2 pure states $\hat{\rho}_{\hat{\alpha}} = \hat{\Omega}_{\hat{\alpha}}$ for which one probability, $P_{\hat{\alpha}}$, vanishes. This yields for Δ^2 the upper bound $m^{-1}[m(m + 1)(m - 1)^2]^{1-m^2}$.

We found that the dynamical processes which afford the best determination of $\hat{\rho}$ are remarkably simple. Indeed, the various types of two-level systems on which experiments are currently performed (NMR, quantum and atomic optics, spintronics) feature Hamiltonians similar to (7) and (10) that optimize the process, with Ising or Heisenberg types of couplings. For instance, the spin-spin interaction between two single-electron quantum dots is usually anisotropic due to spin-orbit coupling or to a lack of symmetry of the host material; see [10] for a recent discussion. Experiments can therefore easily be designed along the above ideas. They will demonstrate that the principle of complementarity, which seems to imply that different measurement devices are needed to fully determine a quantum state, can be bypassed by using an assistant, even completely disordered.

The work of A. E. A. is supported by FOM/NWO. R. B. acknowledges hospitality at the University of Amsterdam, and A. E. A. and Th. M. N. at the CEA-Saclay.

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