Dynamics of Weakly Localized Waves

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We develop a transport theory to describe the dynamics of (weakly) localized waves in a quasi-1D tube geometry both in reflection and in transmission. We compare our results to recent experiments with microwaves and to other theories, such as random matrix theory and supersymmetric theory.

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Localization of waves has always been among the most difficult yet most fascinating topics in the study of wave propagation in disordered media. The first studies dealt with infinite media, showing that localization is always achieved in 1D but that a minimum amount of disorder is required in dimensions larger than 2 [1]. In 3D the critical point is estimated by the Ioffe-Regel criterion $k\ell \approx 1$, with k the wave number and ℓ the mean free path of the waves at a specified frequency [2]. Later studies [3] have considered localization in open media and emphasized the "leakage" through the boundaries — quantified by the conductance — as the basic localization parameter. The Thouless criterion [4] states that "leaky," extended states become localized when the "dimensionless conductance" g is of order one. Most recent studies, both in theory [5] and experiment [6], have emphasized the giant fluctuations in transmission coefficients in the regime $g < 1$, confirming the fundamental importance of the Thouless conductance for all localization phenomena. In the diffuse regime ($g \gg 1$), apart from a factor of order unity, the dimensionless conductance g can be expressed as the ratio of the inverse microscopic level spacing, called the Heisenberg time t_H, and the Thouless time $t_D = (L + 2z_0)^2/\pi^2 D_B$ (with D_B the diffusion constant, L the size of the medium, and $z_0 \sim \ell$ accounting for internal reflection).

A theory for "all localization" does not exist. Important elements should be its capability to describe the transition from the diffuse to the localized regime, notably with regard to leakage and dynamics for any dimensionality, and its flexibility in describing experimental details, such as internal reflection, anisotropic scattering, and absorption. A very complete localization theory is random matrix theory [5]. It can describe the transition from weak to strong localization, scaling, absorption, fluctuations, and, recently, also dynamics [7], but the random matrix theory applies only for low-dimensional systems. Supersymmetric theory [8,9] also has a very general range of applicability but does not always give the necessary physical insight to guide experiments. Finally, the self-consistent theory for localization [10] holds in all dimensions, is able to describe critical behavior around the mobility edge [11], and has a clear generalization for dynamical problems. Its major disadvantages are that it applies only to the field correlation function and not to higher moment statistics, and its failure in the case of broken time-reversal invariance. It is valid on length scales larger than the mean free path and — in the time domain — for times less than the Heisenberg time.

Leakage effects can be studied from the "leakage function" (LF) $P_{T,R}(\alpha)$, defined from the ensemble-averaged, time-dependent transmission (reflection) $I_{T,R}(t)$, according to

$$I_{T,R}(t) = \int_0^\infty d\alpha \exp(-\alpha t) P_{T,R}(\alpha). \qquad (1)$$

Supersymmetric theories [9,12] have predicted strongly nonexponential decay in transmission, even for weakly localized waves ($g \gg 1$) and in quasi-1D typically of the kind "$\exp[-g\ln^2(t/t_H)]$" beyond the Heisenberg time. It is in this regime that a modal picture is appropriate, like in chaotic cavities [13], and that $P_{T,R}(\alpha)$ can be argued to equal the genuine distribution of resonant widths $P(\Gamma)$ of the modes [14] at small Γ. $P(\Gamma)$ has a log-normal behavior at very small Γ attributed to "prelocalized" modes [9], which have become a central issue in the study of random lasers [15]. For α larger than the inverse Heisenberg time (the typical level spacing), the equivalence between $P_{T,R}(\alpha)$ and $P(\Gamma)$ is not established. In particular, $P_{T,R}(\alpha)$ sometimes takes negative values.

For times smaller than the Heisenberg time t_H, supersymmetric theory predicts the transmission to decay like $\exp[-t/t_D + (1/g\pi^2)t^2/t_D^2]$ [9]. This would imply a narrow Gaussian distribution for $P_T(\alpha)$, centered around the average Thouless leakage $1/t_D$ with width $\sim 1/\sqrt{g}$. A recent numerical simulation of wave dynamics in 2D disordered media [16] has shown a similar, roughly quadratic increase of the logarithm of intensity. Chabanov, Zhang, and Genack [17] recently studied weakly localized microwaves in quasi-1D at time scales up to the Heisenberg time, and observed a nonexponential transmission with time of the same type. Another interesting report — coming from random matrix theory [18], and first reported for purely 1D systems [19] — is the $1/t^2$ reflection coefficient for the semi-infinite quasi-1D tube, rather than the familiar $1/t^{3/2}$ decay expected from diffusion theory. This implies that in reflection $P_R(\alpha) \propto \alpha$ for small α.

Transport theory ought to be valid for times less than the Heisenberg time, beyond which a modal picture takes over. The recent developments in theory and experiment call for a transport theory for the dynamics of (weakly) localized waves, and notably for the leakage functions $P_{T,R}(\alpha)$ defined in Eq. (1). This is the subject of the present Letter. First, we show that these functions are broadened by interference effects in a way compatible with observations and supersymmetric theory. Next, we propose both a numerical analysis and an analytical perturbation theory for the weak localization regime $g > 1$, which confirm the conjecture [17] that the time-dependent transmission can be described by a time-dependent diffusion coefficient $D(t)$ decreasing with time. Finally, the calculated dynamic reflection is compared to the random matrix theory result [18].

Constructive interferences can be included into transport theory using the self-consistent theory of localization. In finite, open media this requires the appearance of a dynamical, spatially dependent diffusion constant $D(\mathbf{r}, \Omega)$ [20], which can explain the observed rounding of coherent backscattering of light near the mobility edge [20,21] as well as the non-Ohmic transmission [22]. We will here study the dynamics. Given a short release of energy at the source at time $t = 0$, the central observable is the flux of ensemble-averaged photon energy $I(\mathbf{r}, t)$ at position $\mathbf{r}$ and at time t, with Fourier transform $I(\mathbf{r}, \Omega)$, which we shall continue analytically in the whole complex plane. By causality $I(\mathbf{r}, \Omega)$ is an analytic function in the upper complex sheet $\mathrm{Im}\,\Omega > 0$. For positive times we can change the contour of the inverse Fourier transform with respect to frequency into the negative complex plane. If we assume that simple poles or branch cuts appear only along the negative imaginary axis, we find relation (1) with

$$P_{T,R}(\alpha) = -i\lim_{\epsilon\downarrow 0}[I_{T,R}(\Omega = -i\alpha + \epsilon) - I_{T,R}(\Omega = -i\alpha - \epsilon)]. \quad (2)$$

In the normal diffuse regime only simple poles show up at $\Omega_n = -in^2/t_D$, and $P_{T,R}(\alpha)$ equals an infinite sum of Dirac delta distributions. Purely localized modes would show up as a contribution $\delta(\alpha)$ at zero leakage, but occur only in infinite or closed media. For an open quasi-1D system ($N \gg 1$ transverse modes, length $L \gg \ell$, classical diffusion constant $D_B = v_E\ell/3$, transport mean free path $\ell \gg$ wavelength, and the energy transport velocity v_E) the basic equation is the 1D dynamic diffusion equation for the intensity Green function $C(z, z', \Omega)$,

$$[-i\Omega - \partial_z D(z, \Omega)\partial_z]C(z, z', \Omega) = \delta(z - z'), \quad (3)$$

supplied by the self-consistency condition for the dynamic diffusivity imposed by reciprocity [20],

$$\frac{1}{D(z, \Omega)} = \frac{1}{D_B} + \frac{2}{\xi}C(z, z, \Omega), \quad (4)$$

featuring the length scale $\xi = \frac{2}{3}N\ell$. At the boundaries $z = 0, L$ we impose the usual radiative boundary conditions $C \mp z_0[D(0/L, \Omega)/D_B]\partial_z C = 0$, where $z_0 \sim \ell$ accounts for internal reflection. $I_{T,R}$ is related to C through $I_{T,R}(\Omega) = \mp D(z = L/0, \Omega)\partial_z C(z = L/0, z' = \ell, \Omega)$, where we assume that the incident wave generates an isotropic source at a distance $z' = \ell$ from the surface $z = 0$ [2].

The stationary problem ($\Omega = 0$) can be solved analytically by the substitution $d\tau = dz/D(z, 0)$. This shows that for $L \gg \xi$ the average transmission decays as $\exp(-L/\xi)$, which identifies ξ as the localization length. The diffuse regime $L \ll \xi$ has normal Ohmic transmission with conductance $g \simeq g_0 = \frac{4}{3}N\ell/(L + 2z_0) \simeq 2\xi/L$. These results basically agree with the ones obtained from the Dorokhov-Mello-Pereyra-Kumar (DMPK) equation [5] and supersymmetric theory [23]. Note that when $L \gtrsim \xi$, it is important to discriminate between g_0 and the real conductance g, which can be much smaller by localization effects.

The solution for any complex-valued Ω has to be found numerically by iteration. For $g_0 \gtrsim 0.1$, we found satisfying and unique convergence for all Ω after 10–100 iterations. We have evaluated the leakage function by solving Eqs. (2)–(4) for $\epsilon = 10^{-9}/t_D$ and $\epsilon = 10^{-10}/t_D$ and then using linear extrapolation to find the limit $\epsilon \downarrow 0$. We have also carefully checked the absence of singularities away from the negative imaginary axis. The time-dependent transmission $I_T(t)$ was then obtained from Eq. (1). Absorption can be added, but this will just give rise to a trivial translation of the LF $P_T(\alpha)$ to higher values for α. Following Chabanov, Zhang, and Genack [17] we shall *interpret* any nonexponential decay in terms of a *time-dependent* diffusion constant, in which case the transmission would decay as

$$I_T(t) \sim \exp\left\{-\frac{\pi^2}{(L + 2z_0)^2}\int_0^t dt'\, D(t')\right\}. \quad (5)$$

In Fig. 1 we have compared our calculation for $D(t)$ to experimental results obtained for three different choices for the dimensionless conductance, corresponding to the samples *A*–*C* of Ref. [17]: $g_0 = 9$ (*A*), $g_0 = 7.5$ (*B*), and $g_0 = 4$ (*C*). These values are slightly larger than can be estimated from the data of Ref. [17]. Our transport theory describes the experimental results fairly well for all times below the Heisenberg time $t_H \sim g_0 t_D$. The inset of Fig. 1 shows that the different branches of the leakage function $P_T(\alpha)$ achieve a finite width, though all with finite support. Note that the second branch has a negative value. We have fitted the first, positive branch to a Gaussian distribution with the same average and the same variance, and studied their variation with g_0. The Gaussian distribution leads to a linear decrease of $D(t)$ shortly after the

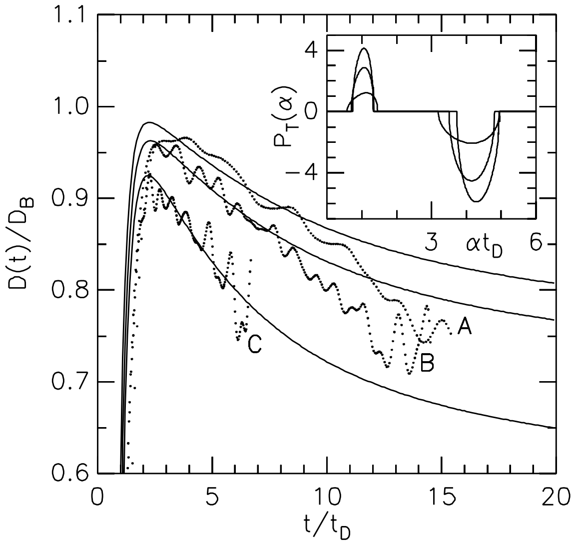

FIG. 1. Time-dependent diffusion constant for wave transmission through a quasi-1D disordered waveguide. Theoretical results (solid lines) are compared to the experimental data of Ref. [17] (dots). Satisfactory agreement between theory and experiment for times t below the Heisenberg time $t_H \sim g_0 t_D$ is obtained by choosing g_0 equal to 9 (sample A), 7.5 (sample B), and 4 (sample C), which are 14% to 33% larger than the experimentally reported values. Oscillations seen in the data are not described by the theory and their exact origin is not clear for the moment [24]. Inset: The leakage function $P_T(\alpha)$ used to obtain the main plot.

diffusion time t_D. Our findings can be summarized by the relation

$$\frac{D(t)}{D_B} = 1 + \frac{A}{g_0} - \frac{B}{g_0}\frac{t}{t_D}, \qquad (6)$$

with $A = 0.15$ and $B = 0.20$.

It is worthwhile to note that Eq. (6) can also be obtained analytically from the first-order perturbation theory in $1/g_0 \ll 1$. We perform the first iteration of Eqs. (3) and (4) starting with $D(z, \Omega) = D_B$ and neglect all terms decreasing faster than $\exp(-t/t_D)$ in the final result for $I_T(t)$. This yields Eq. (6) with $A = 3/(2\pi^2)$ and $B = 2/\pi^2$, which is consistent with numerical results. In the weak localization regime $g = g_0 \gg 1$, supersymmetric theory [9] gives *exactly* the same value for B (orthogonal symmetry), but makes no report of A. Yet, we have noticed that the latter term increases the agreement with experiment considerably. In perturbation theory, B does not depend on the positions of source (z') and receiver (z), contrary to A. The value for $A = 3/(2\pi^2)$ corresponds to transmission ($z' = \ell$, $z = L$), and we find $A = -3/(2\pi^2)$ for a source in the middle of the sample ($z' = L/2$, $z = L$). This perturbation theory can be generalized to higher dimensions provided that $t_D < t < t_H$ and $k\ell \gg 1$. This again yields Eq. (6), with A and B depending on the sample geometry and size. For example, for a point source of waves located in the center of a 2D disordered disk and receiver at the boundary of the disk [16], Eq. (6) reproduces qualitatively the results of Ref. [16].

Our theory assigns no weight to $P_T(\alpha)$ for values smaller than a certain threshold $\alpha^* \approx (1/t_D) \times (1 - 0.8/\sqrt{g_0})$, in strong disagreement with supersymmetric theory [9,12], which predicts a log-normal distribution for small α, caused by prelocalized states that have localization lengths much smaller than the average localization length ξ. Our transport theory is not valid when α is small compared to the average level spacing. It is for this reason that in the localized regime $g < 1$, when the Heisenberg time is smaller than the diffusion time, our theory does not provide a correct description of long-time ($t > t_D$) wave dynamics in transmission.

We will finally study the dynamics in reflection and apply the same procedure to calculate the leakage function $P_R(\alpha)$. For $g \gg 1$ we find a series of clearly separated branches, all *positive* in sharp contrast to transmission, and again with width $\sim 1/\sqrt{g}$. Their maxima typically vary as $\sqrt{\alpha}$, which generates the typically diffuse $1/t^{3/2}$ tail in the time domain. The threshold leakage rate $\alpha^* \sim 1/t_D$ causes an exponential decay at times beyond the diffusion time t_D.

As g decreases, the different branches of $P_R(\alpha)$ start to join when $g_0 \approx 0.5$. For $g_0 \ll 1$, the threshold leakage rate decreases exponentially with g_0, $\ln\alpha^* \sim -1/g_0$, and becomes rapidly very small, implying the disappearance of exponential decay. We will consider a waveguide of length $L \gg \xi$. We find that when $L \gtrsim 20\xi$, $P_R(\alpha)$ has converged to its asymptotic limit at $L \to \infty$. In this limit, $g_0 = 0$ and $t_H = \infty$, and our theory applies at all times. The asymptotic $P_R(\alpha)$ roughly has a square root behavior that is taken over by a *linear* slope for small values of α (see the inset of Fig. 2). The linear law gives rise to the tail $I_R(t) \sim 1/t^2$ in the time domain, as can be seen in Fig. 2. This is consistent with the prediction of Titov and Beenakker using random matrix theory [18]. They have estimated the crossover to occur at a time $t \sim N^2 t_s$ (where $t_s = \ell/v_E$ is the mean free time), again consistent with our findings. We conclude that this interesting dynamical crossover is well captured by the self-consistent transport theory, which, in contrast to the method of Ref. [18], is not limited to the case of $L \gg \xi$ and can be applied to a waveguide of any length and at any time below the Heisenberg time t_H.

In conclusion, we have shown that the dynamics of (weak) localization both in transmission and in reflection of a quasi-1D waveguide can be described by a self-consistent diffusion equation. This theory is not valid beyond the Heisenberg time, and other methods such as those proposed by supersymmetric σ models have to be employed. Stimulated by recent accurate time-resolved experiments on strongly disordered 3D materials close to the mobility edge [25], a future challenge is the application of this theory to 2D and 3D systems for which some

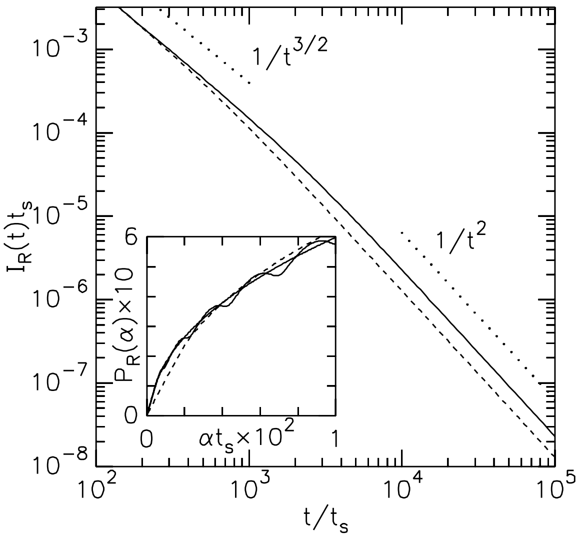

FIG. 2. Time-dependent reflection from a quasi-1D waveguide of length $L \gg \xi$: $N = 20$ (solid line) and $N = 10$ (dashed line). Dotted lines show the slopes $1/t^{3/2}$ and $1/t^2$. Time has been normalized by the mean free time t_s. The curves are obtained by Laplace transforming $P_R(\alpha)$ shown in the inset ($N = 20$, $L = 100\ell$, wavy solid line; $N = 20$, $L \to \infty$, solid line; $N = 10$, $L \to \infty$, dashed line).

perturbational results have already been obtained in this work.

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