Phase Transitions Driven by State-Dependent Poisson Noise

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Nonlinear systems driven by state-dependent Poisson noise are introduced to model the persistence of climatic anomalies in land-atmosphere interaction caused by the soil-moisture dependence of the frequency of rainfall events. It is found that these systems may give rise to bimodal probability distributions, while the state variable randomly persists around the preferential states because of transient dynamics that are opposite to the long-term behavior. Mean-field analysis and numerical simulations of the spatially distributed systems reveal a symmetry-breaking bifurcation for sufficiently strong spatial diffusive couplings and intermediate noise intensities. In such conditions, the initial development of spatial patterns is followed by a stable configuration, selected on the bases of the initial conditions in correspondence of the remnants of the modes of the uncoupled system.

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Noises acting on nonlinear systems can be a source of new phenomena which may qualitatively change the system behavior, eliciting the appearance of bimodality in the steady-state probability density function (pdf), inducing temporal persistence around preferential states, and favoring developments of spatial patterns and symmetry-breaking bifurcations in spatially distributed systems [1–5]. These theoretical findings have provided important applications in physics, engineering, and natural sciences [1,2,5–7].

While the picture is relatively clear for Gaussian noise, the role of other forms of noise has been less investigated. In particular, the case of state-dependent jump (Poisson) noise apparently has never been studied in detail in relation to noise-induced phase transitions. Building upon previous analyses on systems driven by marked Poisson noise [8–14], in which both the jumps and their times of occurrence are random, here we analyze the effect of a state dependence of the noise, especially in terms of the frequency of jumps. As will be seen, the problem is foreboding of rich dynamical behaviors with possible applications in the theory of queues and stochastic reservoirs [10,15] and, especially, in the problem of land-atmosphere interaction [7,12,16].

Consider the stochastic differential equation

$$\frac{ds}{dt} = -\rho(s) + F(t, s), \tag{1}$$

where $\rho(s)$ is a deterministic function of s and $F(t, s)$ is a state-dependent marked Poisson noise

$$F(t, s) = \sum_i y_i \delta(t - t_i), \tag{2}$$

where $\delta(\cdot)$ is the Dirac delta function and the random times $\{t_i\}$ form a point process whose sequence of events takes place according to a nonhomogeneous Poisson process [17], with state-dependent instantaneous rate $\lambda[s(t)]$. The random jumps y_i are described by a state-dependent distribution $b(y; s)$, that gives the probability of jumps of size y starting from the state s. The case of a multiplicative function $g(s)$ acting on the noise term $F(t, s)$ can be brought back to Eq. (1) by changing the jump size to $z_i = g(s)y_i$, with distribution $b'(z; s) = b[y/g(s); s]/g(s)$; thus for an original exponential distribution $b(y)$ with mean $1/\gamma(s)$, the new distribution $b'(y)$ is still exponential with mean $[\gamma(s)g(s)]^{-1}$.

In a spatially lumped description of the terrestrial water balance at the daily time scale, s would represent the relative soil moisture of the active soil layer (i.e., the root zone), $\rho(s)$ the losses due to evapotranspiration and deep infiltration during interstorm periods, and $F(t)$ the jumps in soil moisture by rainfall events. The latter ones may be assumed to partly depend on s because of the so-called precipitation recycling and land-atmosphere interaction. In what follows, having the stochastic soil-moisture dynamics in mind, we will focus on cases where both the deterministic function $\rho(s)$ and the jumps y_i are positive and the process is bounded between 0 and 1. The lower bound at zero is imposed by requiring $\rho(0) = 0$, while the upper bound is ensured by limiting the jumps at $s = 1$ by means of a Dirac delta function at $1 - s$ in the jump distribution [12,13]. Accordingly, an (unbounded) exponential jump distribution with constant mean $1/\gamma$ becomes

$$b(y; s) = \gamma e^{-\gamma y} + \delta(y - 1 + s) \int_{1-s}^{\infty} \gamma e^{-\gamma u} du, \tag{3}$$

for $0 < y \le 1 - s$. This form will be used in what follows.

While [8–13] have studied in detail the steady-state properties of the homogenous case (i.e., with parameters of the Poisson noise independent of s), the only form of state dependence of the noise previously investigated is

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the upper bound in the jump distribution, introduced in [12,13] to account for the saturation conditions in the stochastic soil-moisture dynamics. Here we show that the nonhomogenous problem, yet remaining amenable of analytical developments, may give rise to noise-induced phase transitions. These are qualitatively different from those found for special forms of $\rho(s)$ in [14], where the bimodality was a mere static effect.

Similarly to the case of homogeneous Poisson noise [8–12], the forward Chapman-Kolmogorov equation for state-dependent noise can be written as

$$\frac{\partial p(s,t)}{\partial t} = \frac{\partial}{\partial s}[p(s,t)\rho(s)] - \lambda(s)p(s,t) + \int_0^s \lambda(z)p(z,t)b(s-z;z)dz, \tag{4}$$

where $p(s,t)$ is the time-dependent pdf of s [18]. A relatively simple solution of (4) is possible for steady-state conditions, i.e.,

$$p(s) = \frac{C}{\rho(s)} e^{-\gamma s + \int_s [\lambda(u)/\rho(u)]du}, \tag{5}$$

where C is a normalization constant. Interestingly, the form of Eq. (4) is the same as that of the homogeneous case ($\lambda = \text{const}$) [9,10] and does not depend on the presence of the upper bound, whose effects thus remain entirely embedded in the normalization constant [12,13].

A simple explicit solution for which the appearance of preferential states can be analyzed in detail is offered by the nonhomogeneous Takacs process [8], i.e., with linear decay, $\rho(s) = \beta = \text{const}$, bounded between 0 and 1 and with rate of arrival linearly dependent on s, $\lambda(s) = a + bs$. In this case, the steady-state distribution is a mixed one [8,13,18] with an atom of probability at zero,

$$p(s) = C\left[\frac{1}{\beta} e^{-[\gamma-(a/\beta)]s+(b/2\beta)s^2} + \frac{\delta(s)}{a}\right], \tag{6}$$

with

$$C = \frac{1}{\frac{1}{a} + \sqrt{\frac{\pi}{2b\beta}} e^{-\eta^2}[\text{Erfi}(\eta+b) - \text{Erfi}(\eta)]}, \tag{7}$$

where $\eta = (a - \beta\gamma)/\sqrt{2b\beta}$ and Erfi$(\cdot)$ is the imaginary error function [19]. Figure 1 shows a plot of the pdf of s: when $b \neq 0$, the continuous part of the pdf can be bimodal, with modes at $s = 0$ and $s = 1$. The existence of these preferential states is observed in the time series of s (Fig. 1). Notice that in the hydrologic interpretation this would imply a persistence of dry and wet states in the dynamics of the terrestrial water balance.

The initial transient of the mean trajectory plays an important role in the developments of noise-induced phase transitions [4]. In the present case, the so-called macroscopic equation for the temporal evolution of the mean, $\langle s(t)\rangle$, can be derived from (4) as [20,21]

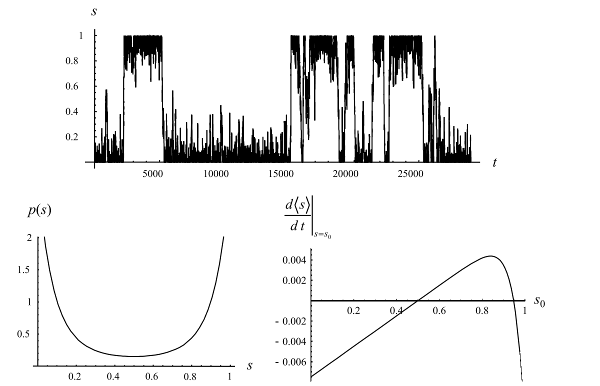

FIG. 1. Nonhomogeneous Takacs process ($a = 0.1$, $b = 0.3$, $\beta = 0.0125$, $\gamma = 20$). Top: Example of time series. Bottom: Continuous part of the steady-state pdf (left) and initial slope of the mean trajectory, Eq. (9), as a function of s_0 (right).

$$\frac{d\langle s(t)\rangle}{dt} = \frac{\langle\lambda(s)\rangle}{\gamma} - \langle\rho(s)\rangle - \frac{1}{\gamma}\int_0^1 \lambda(z)p(z,t)e^{-\gamma(1-z)}dz. \tag{8}$$

Assuming $p(s, t = 0) = \delta(s - s_0)$, the slope of the mean trajectory starting from s_0 is found to be

$$\left.\frac{d\langle s(t)\rangle}{dt}\right|_{s=s_0} = -\rho(s_0) + \frac{\lambda(s_0)}{\gamma}[1 - e^{-\gamma(1-s_0)}]. \tag{9}$$

This is plotted in Fig. 1 for the nonhomogeneous Takacs process: the destabilizing action of the noise in some parts of the s domain is clearly evident.

Following [3,4], the analysis may be extended to a spatially distributed model with diffusive coupling, in which the scalar variable $s_{\mathbf{r}}$ is assumed to evolve over a d-dimensional cubic lattice according to

$$\frac{ds_{\mathbf{r}}}{dt} = F_r(t, s_{\mathbf{r}}) - \rho(s_{\mathbf{r}}) + \frac{D}{2d}\sum_{\mathbf{r}'\in n(\mathbf{r})}(s_{\mathbf{r}'} - s_{\mathbf{r}}), \tag{10}$$

where $\mathbf{r}$ is the vector determining the position of the lattice point, $n(\mathbf{r})$ is the set of $2d$ cells neighbor of $\mathbf{r}$, and D is a diffusion coefficient. The noise terms for each site, $F(t, s_{\mathbf{r}})$, are given by (2) and assumed to be spatially uncorrelated, while $\rho(s_{\mathbf{r}})$ is the same deterministic function for all sites.

Noticing that, once $\rho(s)$ is replaced by $\rho(s_{\mathbf{r}}) - (D/2d)\sum_{\mathbf{r}'\in n(\mathbf{r})}(s_{\mathbf{r}'} - s_{\mathbf{r}})$, Eq. (10) is the same as Eq. (1), the multivariate steady-state pdf can be obtained from (5) by substituting $\rho(s)$ with $\rho(s_{\mathbf{r}}) + D[s_{\mathbf{r}} - E(s_{\mathbf{r}})]$, where $E(y) = \langle s_{\mathbf{r}'}|s_{\mathbf{r}} = y\rangle$ is the steady-state conditional average of s in the neighborhood of lattice point $\mathbf{r}$, given that $s_{\mathbf{r}} = y$. As noticed in [4], this is an exact result, but $E(y)$ is unknown. A useful approximation can be obtained by resorting to the Weiss's mean-field hypothesis (e.g., [4], and references therein), which consists in neglecting the

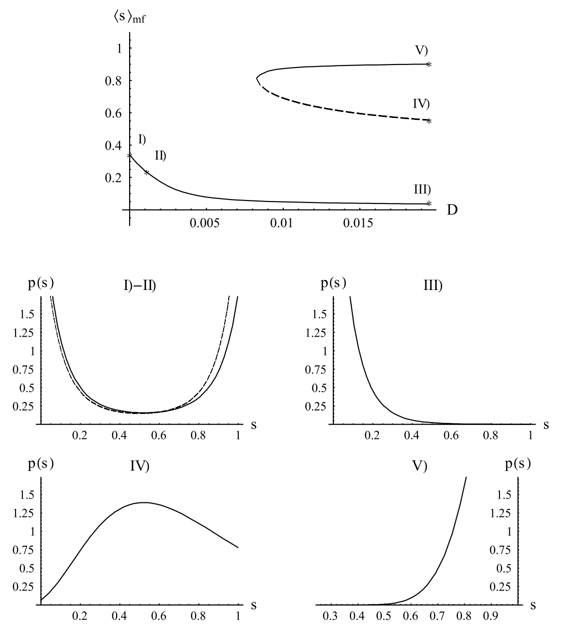

FIG. 2. Mean field hypothesis for the nonhomogeneous Takacs process ($a = 0.1$, $b = 0.3$, $\beta = 0.0125$, $\gamma = 20$). Top: solutions of the self-consistency equation, Eq. (11), as a function of the diffusion coefficient. Bottom: corresponding examples of the continuous part of the steady-state spatial pdf's (the dashed line refers to case I).

spatial correlations and assuming that $E(y) = \langle s \rangle$ (the subscript $\mathbf{r}$ is omitted hereafter). For the system to be self-consistent, $\langle s \rangle$ must be given by

$$\langle s \rangle = \int_0^1 s p(s) ds = f(\langle s \rangle), \tag{11}$$

whose possible multiple solutions may be associated with the breaking of ergodicity and the emergence of phase transitions.

For the nonhomogeneous Takacs process, the solutions of the self-consistency equation are shown in Fig. 2 along with the corresponding long-term spatial pdf's. For low values of D only one steady-state value of $\langle s \rangle$ exists (cases I and II). As the diffusive coupling is increased under constant noise intensity, the system behavior departs from that of the uncoupled case, until a bifurcation (supposedly an imperfect pitchfork one, e.g., [22]) takes place and three solutions appear. The two extreme ones (e.g., cases III and V) are actually observed during the numerical simulations, while the intermediate one (case IV) is never observed and thus presumably unstable. Conversely, if one considers increasing noise intensities [e.g., $\lambda(s)/\gamma$] for a fixed diffusive coupling, the phase transition appears to be reentrant (Fig. 3). As in [3], the ordered phase appears for a window of intermediate noise intensities and is destroyed at high noise levels.

A typical evolution is shown in Fig. 4 for uniform initial conditions $s_0 = 0.55$. Initially, the system is driven towards a bimodal configuration in which high s values

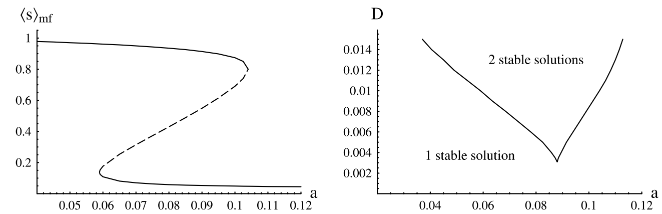

FIG. 3. Nonhomogeneous Takacs process. Left: solutions of the self-consistency equation, Eq. (11), as a function of the noise intensity. The latter, which is equal to $\lambda(s)/\gamma$, is varied by changing a while keeping b, γ, and a/β constant ($b = 0.3$, $\gamma = 20$, $a/\beta = 8$, $D = 0.01$). Right: Phase diagram for the noise-induced phase transition as predicted by the mean-field hypothesis ($a/\beta = 8$, $b = 0.3$, $\gamma = 20$).

(light pixels) are prevalent while the spatial mean is increasing; afterwards, the situation is inverted and the low s values (dark sites) slowly take over (this corresponds to case III of Fig. 2). For initial conditions greater than 0.6, the final configuration is opposite (e.g., case V of Fig. 2). It remains to be clarified whether these final configurations are effectively stable even to exceptional events involving simultaneously the majority of the sites, such as widespread occurrence of jumps or prolonged absence thereof.

The results presented above are relevant for the problem of land-atmosphere interaction. Recent hydrologic investigations have shown that rainfall representation as a marked Poisson process provides a realistic description of the precipitation component in the dynamics of the soil water balance at the daily time scale [12,13,16]. Thus the soil-moisture dynamics may be described by a stochastic system of the type of Eq. (1) driven by homogeneous Poisson noise and nonlinear $\rho(s)$ accounting for evapotranspiration and deep infiltration. However, when the

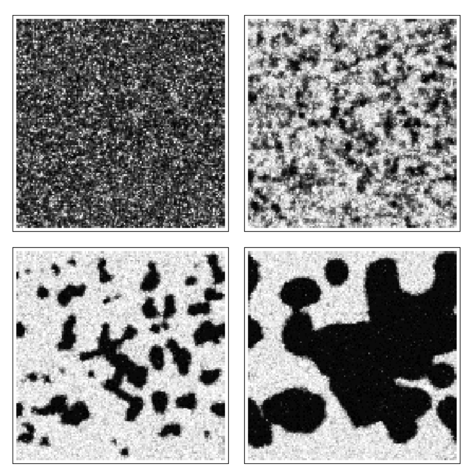

FIG. 4. Evolution of the spatially distributed system with $D = 0.05$ for the process with linear decay ($a = 0.1$, $b = 0.3$, $\beta = 0.0125$, $\gamma = 20$), starting from uniform initial condition $s_0 = 0.55$. Snapshots at times 30, 150, 500, 3000, using a 128×128 grid with periodic boundary conditions.

water balance is analyzed at the continental scale, possible feedbacks (either due to local precipitation recycling or to interactions in the soil-atmosphere energy balance) may become important [7] and induce a soil-moisture dependence of precipitation. To this regard, recent analyses [23] of field data indicate an increase of the rainfall frequency at high soil-moisture levels. A simple representation of this phenomenon would entail a state-dependent rainfall frequency $\lambda(s)$, i.e., function of the relative soil moisture, s. While a detailed theoretical analysis and comparison with field data will be presented elsewhere [23], a hydrologic interpretation of the processes described before already suggests the possibility of temporal persistence of dry and wet states in the terrestrial water balance and the appearance of spatio-temporal patterns in soil-moisture and precipitation dynamics at the continental scale.

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