# Capacity of Oscillatory Associative-Memory Networks with Error-Free Retrieval

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Networks of coupled periodic oscillators (similar to the Kuramoto model) have been proposed as models of associative memory. However, error-free retrieval states of such oscillatory networks are typically *unstable*, resulting in a near zero capacity. This puts the networks at disadvantage as compared with the classical Hopfield network. Here we propose a simple remedy for this undesirable property and show rigorously that the error-free capacity of our oscillatory, associative-memory networks can be made as high as that of the Hopfield network. They can thus not only provide insights into the origin of biological memory, but can also be potentially useful for applications in information science and engineering.

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The celebrated Hopfield model of associative memory [1] has provided fundamental insights into the origin of neural computations and has since stimulated much interest [2]. In this model, neurons in the network assume discrete values (e.g., $+1$ and $-1$) and a set of patterns is stored such that when a new pattern is presented, the network responds by producing a stored pattern that most closely resembles the new pattern. Of interest is then the capacity, the maximum number of patterns per neuron that the network can "memorize." If a small amount of error is allowed in the retrieval of patterns, the capacity of the classical Hopfield network can be shown to be 0.138 [2,3]. For error-free retrieval of patterns, however, the capacity is generally reduced and is proportional to $1/\log n$, where $n$ is the number of neurons in the network [1]. The physical significance of Hopfield's work lies in his proposal of the energy function and his idea that memories are *dynamically stable* attractors, naturally bringing concepts and tools from statistical and nonlinear physics into neuro- and information sciences as well as engineering.

Recent empirical findings in neuroscience [4,5] suggest that synchronous firing of specific neurons is ubiquitous in the brain. These have stimulated studies of models of *oscillatory* associative memory based on temporal coding of information [6–11]. Such models typically consist of coupled periodic oscillators interacting with each other according to a Hebbian rule, and the patterns are stored as phase-locked oscillations. A paradigm for coupled periodic oscillators is Kuramoto's model [12], which is relevant to many physical and biological phenomena [13]. The oscillatory models of associative memory can be regarded as a generalization of Kuramoto's model and typically take the form

$$\dot{\theta}_i = \omega_i + \sum_{j=1}^{n} C_{ij}\sin(\theta_j - \theta_i + \psi_{ij}), \qquad i = 1, \ldots, n, \quad (1)$$

where each periodic oscillator is dynamically described by a phase variable $\theta_i(t)$, $C_{ij}$ is the coupling matrix, and $\psi_{ij}$ is the synaptic phase delay. An advantage of this type of model is that it can in principle be implemented using a variety of physical devices including phase-locked loop circuits [14], lasers [15], and MEMS resonators [16].

For pattern retrieval with small errors, a standard mean-field treatment gives that the capacity of the Kuramoto-type network described by (1) is about one-third that of the Hopfield network [6–11]. Surprisingly, error-free retrieval solutions appear to be typically *unstable* regardless of the network size, as long as the number of memorized patterns exceeds two [8]. Thus the error-free capacity of the network (1) is $2/n$, which is much smaller than that of the Hopfield network. This implies that this type of network is not suitable for applications that require error-free retrieval of stored patterns. The aim of this Letter is to devise a scheme to overcome this difficulty. In particular, we introduce a second-order mode of strength $\varepsilon$ in the coupling function. We then prove rigorously that the error-free capacity of the improved network is at least $2\varepsilon^2/\log n$. Note that the capacity scales with $n$ in the same manner as in the Hopfield network, but it can be enhanced by increasing $\varepsilon$. Nonetheless, increasing $\varepsilon$ also tends to stabilize solutions encoding patterns other than the stored (desirable) ones, meaning that if $\varepsilon$ is too large, the solution for every possible pattern can become stable, and the system cannot distinguish the memory patterns from others. We are able to argue, however, that the network can function as effective associative memory for a finite range of $\varepsilon$. Thus, our modified oscillatory networks can be used as error-free, associative-memory devices with similar capacity to that of the classical Hopfield network.

We consider a family of oscillator networks with the coupling term $C_{ij}\sin(\theta_j - \theta_i + \psi_{ij})$ in the model (1) replaced by

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$$\Gamma_{ij}(\theta_j - \theta_i) = C_{ij}\sin(\theta_j - \theta_i) + \frac{\varepsilon}{n}\sin 2(\theta_j - \theta_i), \quad (2)$$

where the new term $\frac{\varepsilon}{n}\sin 2(\theta_j - \theta_i)$ corresponds to a second-order Fourier mode and $\varepsilon$ is a parameter characterizing the strength of this term. For this model to describe an associative memory, the elements of the coupling matrix are given by Hebb's learning rule: $C_{ij} = \frac{1}{n}\sum_{\mu=1}^{p}\xi_i^\mu\xi_j^\mu$, where $\xi^\mu = (\xi_1^\mu, \ldots, \xi_n^\mu)^T$ ($\xi_i^\mu = \pm 1$, $\mu = 1, \ldots, p$, $i = 1, \ldots, n$) denotes a set of $p$ patterns to be memorized. We write $p_n$ for the number of patterns to be memorized in a network of size $n$. Patterns are chosen randomly, so $\xi_i^\mu$ are independent and identically distributed random variables with $P(\xi_i^\mu = 1) = P(\xi_i^\mu = -1) = 1/2$. We focus on the case for which the natural frequencies of all oscillators are equal (say, $\omega_i = \omega$). After the change of variable $\theta_i \to \theta_i + \omega t$, the equations of motion become

$$\dot{\theta}_i = \sum_{j=1}^{n} C_{ij}\sin(\theta_j - \theta_i) + \frac{\varepsilon}{n}\sum_{j=1}^{n}\sin 2(\theta_j - \theta_i), \quad (3)$$

where $i = 1, 2, \ldots, n$. These equations are invariant under translation by a constant, implying that there is at least one direction in the phase space in which any solution is neutrally stable. Equations (3) possess $2^n$ fixed-point solutions, corresponding to all possible binary patterns of length $n$. Let $\eta = (\eta_1, \ldots, \eta_n)^T$ be an $n$-dimensional vector of 1's and $-1$'s representing one of those binary patterns. There is a unique (up to constant translation) fixed-point solution corresponding to the pattern $\eta$, which is characterized by $|\theta_i - \theta_j| = 0$ if $\eta_i = \eta_j$ and $|\theta_i - \theta_j| = \pi$ if $\eta_i \neq \eta_j$. We denote this solution by $\theta(\eta)$. In the original coordinates, they are *phase-locked* oscillatory solutions, in which binary patterns are encoded in the locked phase deviations of the oscillators. An example of such a solution is shown schematically in Fig. 1.

The symmetry of the connection matrix $C$ ensures that (3) can be written as a gradient system with the Lyapunov (energy) function

$$L(\theta; \varepsilon, C) = -\frac{1}{2}\sum_{i,j=1}^{n} C_{ij}\cos(\theta_i - \theta_j) - \frac{\varepsilon}{4n}\sum_{i,j=1}^{n}\cos 2(\theta_i - \theta_j). \quad (4)$$

Thus, any solution will eventually converge to a fix point of the system located at a local minimum of the energy function (4). Using Hebb's learning rule for $C$, the energy per oscillator can be expressed as $\bar{L}(\theta; \varepsilon, \Xi) \equiv \frac{1}{n}L(\theta; \varepsilon, \Xi) = -\frac{1}{2}\sum_{\mu=1}^{p} m_\mu^2 - \frac{\varepsilon}{4}q^2$, where $\Xi = (\xi^1, \ldots, \xi^p)$, the order parameters $m_\mu = |\frac{1}{n}\sum_{j=1}^{n}\xi_j^\mu e^{i\theta_j}|$ for $\mu = 1, \ldots, p$, and $q = |\frac{1}{n}\sum_{j=1}^{n} e^{2i\theta_j}|$. The parameter $m_\mu$ is called the *overlap* and measures the closeness of the solution to the memory pattern $\xi^\mu$, and $q$ measures the closeness of the solution to its nearest binary pattern. The second term in $\bar{L}(\theta; \varepsilon, \Xi)$ is necessary since a minimum of the first term is typically located near but off the fixed point corresponding to one of the patterns $\xi^\mu$. The second term always has local minima of the same depth at all fixed points representing binary patterns, and thus, combined with the first term, ensures that the energy minima are located precisely at the memorized patterns. It is worth noting at this point that the overlaps $m_\mu$ for solutions $\theta(\eta)$ coincide with those for the Hopfield model, i.e., $m_\mu = |\sum_j \xi_j^\mu \eta_j / n|$. Moreover, since $q = 0$ for these solutions, the energy levels $\bar{L}(\eta, \Xi) \equiv L[\theta(\eta); \varepsilon, \Xi]/n$ do not depend on $\varepsilon$ and are identical to the energy per spin in the Hopfield model.

We now give general stability results that hold for any solution corresponding to a binary pattern, for a finite $n$. The Jacobian matrix of Eq. (3) evaluated at $\theta = \theta(\eta)$ is $(2\varepsilon/n)E - 2\varepsilon I + J$, where $E$ is the $n \times n$ matrix of ones, $I$ is the $n \times n$ identity matrix, and $J$ is defined componentwise by $J_{ij} = C_{ij}\eta_i\eta_j - \delta_{ij}\sum_{k=1}^{n} C_{ik}\eta_i\eta_k = \frac{1}{n}\sum_{\mu=1}^{p}\xi_i^\mu\xi_j^\mu\eta_i\eta_j - \frac{\delta_{ij}}{n}\sum_{k=1}^{n}\sum_{\mu=1}^{p}\xi_i^\mu\xi_k^\mu\eta_i\eta_k$. The stability of the solution $\theta(\eta)$ is determined by the eigenvalues of the Jacobian matrix. Since it is symmetric, all eigenvalues are real. A solution is stable iff all eigenvalues are negative. Let $\lambda_{\max}(J)$ denote the maximum eigenvalue of the matrix $J$. It can then be shown [17] that the solution of (3) encoding the pattern $\eta$ is asymptotically stable if $\lambda_{\max}(J) < 2\varepsilon$, and unstable if $\lambda_{\max}(J) > 2\varepsilon$. Figure 2 shows sample distributions (over different choices of $\xi^\mu$) of $\lambda_{\max}(J)$ for three types of $\eta$: a memory pattern, a memory pattern with single-bit error, and a random pattern. Notice that even for the memory pattern it is always positive, indicating that the corresponding solution is unstable for $\varepsilon = 0$. In fact, our numerics with various combinations of $n$ and $p$ suggest that the same is true for any $n$ and any $p > 2$ [18]. This is consistent with the observation in Ref. [8]. The role of the second-order mode in the coupling function that we have introduced can now be understood: It shifts

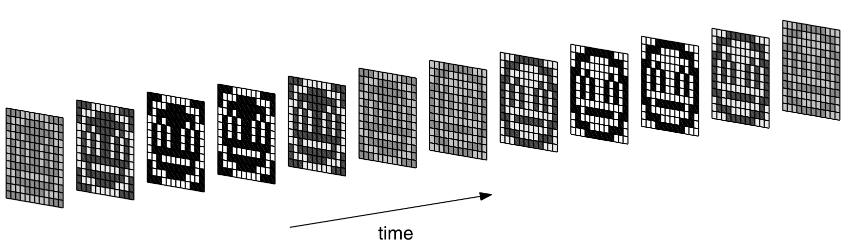


FIG. 1. A pattern encoded in the phase deviation among the oscillators. Each cell representing an oscillator is painted in gray scale according to its phase.

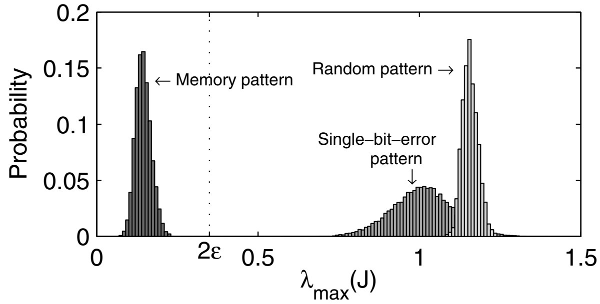


FIG. 2. The distribution of the maximum eigenvalue $\lambda_{\max}(J)$ for three types of solutions: a memory pattern, a memory pattern with single-bit error, and a random pattern. We chose each bit of each memory pattern to be $\pm 1$ at random with equal probabilities. The parameters of the system were $n = 1000$ and $p = 10$. Note that $\lambda_{\max}(J) = 2\varepsilon$ is the borderline of stability.

the eigenvalues by $2\varepsilon$ for each solution corresponding to a pattern.

We next calculate the capacity for error-free retrieval solutions: $\eta = \xi^\nu$ $(\nu = 1, \ldots, p)$. Letting $\eta = \xi^\nu$ for a fixed $\nu$ in the expression for $J$ above yields $J = S + D - \{1 + [(p-1)/n]\}I$, where $S_{ij} = \frac{1}{n}\sum_{\mu=1}^{p} \xi_i^\mu \xi_i^\nu \xi_j^\mu \xi_j^\nu$ and $D_{ij} = -\frac{\delta_{ij}}{n}\sum_{\mu\neq\nu}\sum_{k\neq i} \xi_i^\mu \xi_i^\nu \xi_k^\mu \xi_k^\nu$. The stabilities of these solutions are then determined by $\lambda_{\max}(J)$, whose statistical behavior in the limit $n \to \infty$ depends on the rate of growth of $p = p_n$. We address the question of the capacity in this limit by asking how large $p_n$ can be as a function of $n$.

First consider the case $\varepsilon = 0$. As we have mentioned, the solutions corresponding to memory patterns appear to be unstable for any combination of $p > 2$ and $n$. Here we present an argument that supports this claim. The condition for instability is $\lambda_{\max}(J) > 0$ in this case. A lower bound for $\lambda_{\max}(J)$ can be obtained by the variational principle: $\lambda_{\max}(J) \geq x^T Jx/x^T x$, $\forall x \in \mathbb{R}^n$. By choosing $x$ to be the normalized eigenvector of $S$ associated with the eigenvalue $\lambda_{\max}(S)$, we obtain $\lambda_{\max}(J) \geq \lambda_{\max}(S) + (x^T Dx/x^T x) - 1 - [(p-1)/n]$. Taking averages on both sides, we obtain $\langle\lambda_{\max}(J)\rangle \geq \langle\lambda_{\max}(S)\rangle + \langle x^T Dx/x^T x\rangle - 1 - [(p-1)/n] \approx 2\sqrt{p/n} > 0$. Here we used the approximation, valid for $1 \ll p \ll n$, that the eigenvector $x$ and the components of $D$ are nearly independent, which leads to $\langle x^T Dx/x^T x\rangle \approx 0$. We also used $\langle\lambda_{\max}(S)\rangle \approx 1 + \frac{p}{n} + 2\sqrt{p/n}$, which holds if $n$ and $p$ are both large [19]. That the average value of $\lambda_{\max}(J)$ is bounded away from zero indicates that the memory patterns are unstable for $\varepsilon = 0$.

Our main result for the general case with finite $\varepsilon$ can be summarized in the following theorem [19].

**Theorem:** *Let* $\overline{\alpha} = \overline{\lim}_n[(p_n \log n)/n]$, $\underline{\alpha} = \underline{\lim}_n[(p_n \log n)/n]$, *and* $\varepsilon > 0$. *If* $\overline{\alpha} < 2\varepsilon^2$, *then the solution corresponding to* $\xi^\nu$ *is asymptotically stable with probability one in the limit* $n \to \infty$. *If* $\underline{\alpha} > (1 + 2\varepsilon)^2/2$, *then it is unstable with probability tending to one as* $n \to \infty$.

In other words, as $n$ becomes large, the condition $p_n/n < 2\varepsilon^2/\log n$ guarantees stability, while the condition $p_n/n > (1 + 2\varepsilon)^2/(2\log n)$ guarantees instability of error-free retrieval solutions. In particular, if $p_n = cn$ for some constant $c$, then $\underline{\alpha} = \infty$ and therefore the solution is unstable no matter how large $\varepsilon$ is.

For random patterns for which each $\eta_i$ is chosen randomly and independently to be $\pm 1$ with equal probability, i.e., $\eta$ is chosen in exactly the same fashion as $\xi^\mu$, it is straightforward to show that $\langle\lambda_{\max}(J)\rangle \geq 1 - 1/n$. Actually, we can show [19] that a stronger result holds: If $p_n/n^2 \to 0$ as $n \to \infty$, then $P[\lambda_{\max}(J) \leq 1 - \delta] \to 0$ as $n \to \infty$ for every $\delta > 0$, i.e., $\lambda_{\max}(J) \gtrsim 1$ for $n \gg 1$. Hence, the network is capable of distinguishing the memory patterns from most of the other patterns, provided that $\varepsilon < 1$ [20].

Having established the local stability of a solution, we address the issue of global stability. The existence of the energy function $L(\theta; \varepsilon, \Xi)$ in (4) ensures that any solution of the system converges to a phase-locked solution as $t \to \infty$. On the other hand, the local stability of the solutions representing the memory patterns means that there is an open basin of attraction for each of these solutions. How large are these basins? That is, how close does the initial condition need to be to a memory pattern, in order for the network to evolve into the phase-locked state that encodes that pattern?

To quantify the size of these basins, we look at the relationship between the overlaps for initial and final $\theta$. Recall that overlap of one corresponds to zero distance, while overlap of zero indicates that patterns are as far apart as possible [21]. Figure 3 shows typical plots of the (average) final overlap versus the initial overlap. Each plot shares a general feature: As the initial overlap decreases from one, the final overlap stays approximately

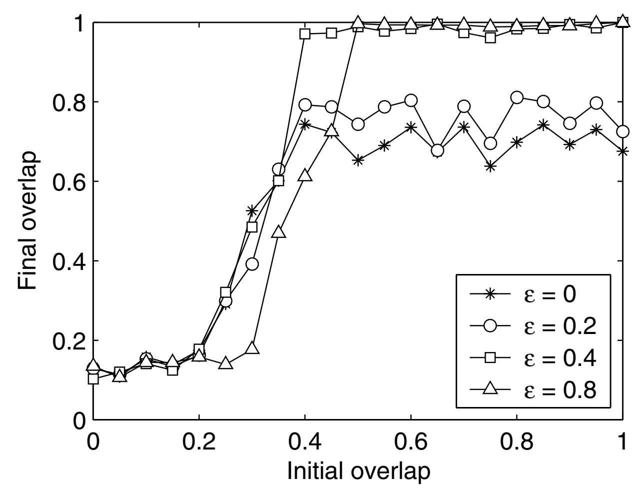


FIG. 3. The final overlap after 1000 time units as a function of the initial overlap for $n = 1000$, $p = 40$, and different values of $\varepsilon$ $( = 0, 0.2, 0.4, 0.8)$. The final overlap is averaged over the results from 10 random initial conditions with the same initial overlap.

constant until the initial overlap reaches the critical value, after which the final overlap decreases sharply. The critical initial overlap appears to be around 0.4 for $\varepsilon = 0, 0.2, 0.4$ and around 0.5 for $\varepsilon = 0.8$. This critical value of initial overlap marks the boundary of the basin of attraction of the memory pattern solution in question, while the value of the final overlap for initial overlaps above the critical value represents the quality of the retrieval process. Figure 3 illustrates two general trends with respect to $\varepsilon$, which are also observed for other typical combinations of $n$ and $p$: (i) As $\varepsilon$ increases from zero, the retrieval error decreases (larger final overlap) until it becomes zero (final overlap is one). Zero error seems to be achieved at a critical value $\varepsilon = \varepsilon_1$, above which the stability condition for the averaged $\lambda_{\max}(J)$ is satisfied. (ii) As $\varepsilon$ increases from zero, the critical value of initial overlap (the size of basin) does not change much until another critical value $\varepsilon = \varepsilon_2$ above which it increases (the basin shrinks). The transition at $\varepsilon = \varepsilon_2$ appears to correspond to the point at which the first solution other than that of a memory pattern becomes stable. Thus, in the range $\varepsilon_1 < \varepsilon < \varepsilon_2$, the optimal performance as associative memory is achieved: No error occurs in the retrieval process and no solution other than that of memory patterns is stable, leading to maximal size of the basin of attraction for the memory pattern solutions.

In summary, we have presented an analysis of the local stability of the error-free memory pattern solutions for a new type of oscillatory model of associative memory. Our model includes an extra, second-order Fourier mode in the coupling function, which enables us to control the stability of the solutions for all patterns and to distinguish the memory pattern from others by their stabilities. The model is also closely related to the cumulative distribution function of spikes in neural networks [22]. The capacity of our model turns out to follow the same scaling with the number of neurons as in the case of the classical Hopfield model, but with a prefactor that depends on the relative strength of the second-order mode. Our conclusion is that, with a simple modification, oscillatory networks of associative memory based on phase locking with a Hebbian connection scheme are capable of performing almost as well as the Hopfield network.

Our model can be modified to allow storage of patterns with $n_s > 2$ symbols. Similar stability results should follow in a straightforward manner simply by replacing the second term of the coupling function with $(\varepsilon/n)\sin[n_s(\theta_j - \theta_i)]$. We also note that the inclusion of the second term in the coupling function does not change the locality of the interactions between neurons [23]. Our coupling function can in principle be implemented using a known electronic circuitry, and thus it would be feasible to implement the entire network as a network of phase-locked loops.

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