Conductance of a Quantum Wire in the Wigner-Crystal Regime

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We study the effect of Coulomb interactions on the conductance of a single-mode quantum wire connecting two bulk leads. When the density of electrons in the wire is very low, they arrange in a finite-length Wigner crystal. In this regime the electron spins form an antiferromagnetic Heisenberg chain with an exponentially small coupling J. An electric current in the wire perturbs the spin chain and gives rise to a temperature-dependent contribution of the spin subsystem to the resistance. At low temperature $T \ll J$ this effect is small, and the conductance of the wire remains close to $2e^2/h$. At $T \gg J$ the spin effect reduces the conductance to e^2/h.

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Experiments with short one-dimensional (1D) conductors (quantum wires) have demonstrated [1] that their conductance is quantized in units of $2e^2/h$. The universality of this result is readily understood in the model of noninteracting electrons. It is remarkable, however, that in most experiments observing the quantization of conductance the Coulomb interactions between electrons are not weak, $(na_B)^{-1} \gtrsim 1$. (Here n is the electron density, $a_B = \varepsilon\hbar^2/me^2$ is the Bohr radius, ε is the dielectric constant, and m is the electron effective mass.) Electron-electron interaction in a 1D system is expected to lead to the formation of a Luttinger liquid with properties very different from those of the noninteracting Fermi gas.

The conductance of an infinite Luttinger liquid was studied by Kane and Fisher [2], who found that it does depend on the interactions. In particular, in the case of repulsive interactions the conductance is below the universal value of $2e^2/h$. The discrepancy between the theory [2] and experiments [1] is usually attributed to the fact that in order to measure conductance of a quantum wire, it has to be connected to two-dimensional Fermi-liquid leads. As electrons move from the wire into the leads, the interactions between them are gradually reduced to zero. At low frequency $\omega \ll v_F/L$ the conductance of such a system is dominated by the leads, and the universal value $2e^2/h$ is restored [3].

The key assumption leading to this result is the applicability of the Luttinger liquid description of the interacting electron system in a quantum wire. In this Letter we show that if the interactions are strong, $(na_B)^{-1} \gg 1$, the Luttinger liquid picture remains valid only at exponentially low temperatures, and we study the corrections to the quantized conductance at higher temperatures.

The low-energy properties of an interacting 1D electron system are most conveniently described by the bosonization approach. In the case of weak interactions the Hamiltonian of the system can be presented [4] as

$$H = H_\rho + H_\sigma, \tag{1}$$

where

$$H_\rho = \int \frac{\hbar u_\rho}{2\pi}[\pi^2 K_\rho \Pi_\rho^2 + K_\rho^{-1}(\partial_x \phi_\rho)^2]dx, \tag{2}$$

$$H_\sigma = \int \frac{\hbar u_\sigma}{2\pi}[\pi^2 K_\sigma \Pi_\sigma^2 + K_\sigma^{-1}(\partial_x \phi_\sigma)^2]dx + \frac{2g_{1\perp}}{(2\pi\alpha)^2}\int \cos[\sqrt{8}\phi_\sigma(x)]dx. \tag{3}$$

Here the fields ϕ_ρ, Π_ρ and ϕ_σ, Π_σ describe the excitations of the charge and spin modes, respectively, and satisfy the bosonic commutation relations: $[\phi_\nu(x), \Pi_{\nu'}(x')] = i\delta_{\nu\nu'}\delta(x - x')$. Parameters K_ρ, K_σ, and $g_{1\perp}$ are determined by the interactions between electrons, u_ρ and u_σ are the velocities of propagation of spin and charge excitations, and α is a short-distance cutoff.

An important feature of the Hamiltonian (1) is the separation of the charge and spin variables [5]. This property is preserved even when bias is applied to the wire, because the electric potential couples only to the charge density, and does not excite the spin modes. As a result, the relatively complicated form (3) of the Hamiltonian H_s is not expected to affect the conductance of the quantum wire.

The derivation [4] of the bosonized Hamiltonian assumes weak interactions between electrons. On the other hand, the form (1)–(3) of the Hamiltonian is universal; i.e., with the appropriate choice of the parameters $K_{\rho,\sigma}$, $u_{\rho,\sigma}$, and $g_{1\perp}$, it is expected to describe the low-energy properties of 1D electron liquids with arbitrarily strong interactions. It will be instructive to obtain the Hamiltonian (1)–(3) from the model of fermions with strong Coulomb interactions, and to estimate the values of the parameters entering Eqs. (2) and (3).

Assuming a quadratic energy spectrum of electrons, $\epsilon(p) = p^2/2m$, one concludes that at low density $n \ll a_B^{-1}$ the potential energy of the electron repulsion $\sim e^2 n/\varepsilon$ is much larger than their kinetic energy $\sim \hbar^2 n^2/m$. Thus, to first approximation one can view the electrons in a quantum wire as a Wigner crystal of particles repelling

each other with strong Coulomb forces. If the Coulomb interaction is screened at large distances by a metal gate parallel to the wire, the density excitations of the Wigner crystal (plasmons) are acoustic waves. Thus the low-energy density excitations of this system must be described by the Hamiltonian of the form (2). The plasmon velocity in a one-dimensional Wigner crystal at a distance d from the gate is $u_\rho = \sqrt{e^2 n/mC}$, where $C = \varepsilon/[2\ln(\zeta n d)]$ is the capacitance per unit length between the crystal and the gate [6]. (Here ζ is a numerical coefficient determined by the geometry of the gate [7].) Parameter K_ρ can be found by comparing the second term in the integrand of Eq. (2) with the charging energy per unit length $E_C = e^2\delta n^2/2C$, where $\delta n(x)$ is the deviation of the density of electrons from its mean value. The bosonization procedure used in the Hamiltonian (2) identifies $\delta n = -(\sqrt{2}/\pi)\partial_x\phi_\rho$. Comparing E_C with the second term in Eq. (2), one then finds $K_\rho = (\pi\hbar/2)\sqrt{nC/me^2}$. Substituting the above expression for C, we summarize

$$u_\rho = \frac{v_F}{K_\rho}, \qquad K_\rho = \frac{\pi}{2}\sqrt{\frac{na_B}{2\ln(\zeta n d)}}, \tag{4}$$

where the parameter $v_F \equiv \pi\hbar n/2m$ is defined as the Fermi velocity of a noninteracting electron gas of density n. As expected, at $(na_B)^{-1} \gg 1$ strong Coulomb interactions result in $K_\rho \ll 1$ and $u_\rho \gg v_F$.

In the limit of strong coupling the energy of a Wigner crystal does not depend on the spins of electrons, as the particles localized near their equilibrium positions can be viewed as distinguishable. This picture is violated if one accounts for the overlap of the wave functions of neighboring electrons. The overlap is due to the possibility of tunneling through the potential barrier $e^2/\varepsilon r$ separating the electrons. To first order in tunneling, the overlap occurs only between the nearest neighbors, and one expects the coupling of the spins to be described by the Hamiltonian of a Heisenberg spin chain:

$$H_\sigma = \sum_l J\boldsymbol{S}_l \cdot \boldsymbol{S}_{l+1}. \tag{5}$$

Since the ground state of a system of interacting fermions in one dimension cannot be spin polarized [8], the exchange must have antiferromagnetic sign, $J > 0$.

A crude estimate of the exchange constant J can be obtained by calculating the amplitude of tunneling through the Coulomb barrier $e^2/\varepsilon r$ separating two neighboring electrons in the WKB approximation. Placing the turning points at $r = \pm n^{-1}$ and using the reduced mass $m/2$, one finds

$$J \sim E_F \exp\left(-\frac{\eta}{\sqrt{na_B}}\right) \tag{6}$$

with the numerical coefficient $\eta = \pi$. (Here $E_F \equiv \pi^2\hbar^2 n^2/8m$ is defined as the Fermi energy of a noninteracting electron gas of density n.) A more careful calculation [9] implies $\eta \approx 2.9$.

The Hamiltonian of the Heisenberg spin chain (5) can be rewritten in terms of spinless fermion operators a_l and $a_l^\dagger$ with the help of the Jordan-Wigner transformation

$$S_l^z = a_l^\dagger a_l - \frac{1}{2}, \quad S_l^x + iS_l^y = a_l^\dagger \exp\left(i\pi\sum_{j=1}^{l-1} a_j^\dagger a_j\right). \tag{7}$$

To study the low-energy properties of the spin chain, one can bosonize the fermion operators a_l and $a_l^\dagger$. As a result the Hamiltonian H_σ takes the form (3); see, e.g., Ref. [4]. The velocity u_σ of the spin excitations is easily deduced from the spectrum [10] of the isotropic Heisenberg model,

$$u_\sigma = \frac{\pi J}{2\hbar n}. \tag{8}$$

The sine-Gordon perturbation in Eq. (3) is marginally irrelevant, i.e., the coupling constant $g_{1\perp}$ scales to zero at low energies. At the same time the parameter K_σ scales to 1, as required by the SU(2) symmetry of the problem [4,11].

It is important to stress that the bosonized form (3) of H_σ is appropriate only at low temperatures, $T \ll J$. In the following we are also interested in the temperature dependence of the conductance at $T \sim J$, and thus we use the form (5). Interestingly, the dynamics of spin and charge modes are still completely separated, as the operators (2) and (5) commute. We now show that this symmetry is violated when the wire is connected to Fermi-liquid leads.

Following Ref. [3], we model the leads attached to the quantum wire by two semi-infinite sections of noninteracting electron gas. To this end we assume that the 1D electron density $n(x) = n$ near $x = 0$, and gradually grows to a very high value $n_\infty \gg a_B^{-1}$ at $x \to \pm\infty$. Assuming that the length scale L of the dependence $n(x)$ is large compared to the distance between electrons, one can neglect the backscattering caused by the inhomogeneity, and describe the charge excitations by the bosonized Hamiltonian (2) with position-dependent parameters $u_\rho(x)$ and $K_\rho(x)$. In addition, the coupling constant J in the Hamiltonian (5) now depends on l due to the density dependence (6) of the exchange interaction. The exchange constant $J[l]$ is determined by the density $n(x_l)$ at the position of the lth electron.

Now let us consider the quantum wire in the regime when an electric current $I = I_0\cos\omega t$ passes through it. We are interested in the dc limit $\omega \to 0$, and can thus assume that all electrons move in phase. The charge transferred through any point in the wire is $Q = I_0\omega^{-1}\sin\omega t$, so at moment t the lth electron has shifted to the position $l + I_0(e\omega)^{-1}\sin\omega t$. Thus, the Hamiltonian of the spin chain takes the form

$$H_\sigma = \sum_l J[l + q(t)]\boldsymbol{S}_l \cdot \boldsymbol{S}_{l+1}, \qquad q(t) = \frac{I_0}{e\omega}\sin\omega t. \tag{9}$$

One can view current $I = e\dot{q}$ as an excitation of the charge mode ϕ_ρ and substitute the appropriate relation

$$q(t) = \frac{\sqrt{2}}{\pi}\phi_\rho(0,t) \tag{10}$$

into the Hamiltonian (9). Therefore the spin modes are coupled to the charge ones and should affect the conductance of the wire.

In the following it is more convenient to treat the current $I(t) = I_0\cos\omega t$ as an external parameter. This approach corresponds to the experiment with the wire connected to a current source. We evaluate the energy W dissipated in the device in unit time in the limit of small I_0 and ω. The dc resistance is found from the Joule heat law $W = \frac{1}{2}I_0^2 R$. The advantage of this approach is that the spin and charge modes are coupled only through the external parameter $I(t)$, so W is a sum of two independent contributions of the charge and spin modes. Thus, the resistance of the wire is the sum of two terms $R = R_\rho + R_\sigma$, which can be evaluated by considering the Hamiltonians H_ρ and H_σ separately.

The result $G = 2e^2/h$ for the conductance of a quantum wire found in Ref. [3] amounts to the calculation of the resistance R_ρ, as the spin modes were ignored. Thus, one expects to find $R_\rho = h/2e^2$. Let us outline the derivation of this result in the approach where the current $I(t)$ through the wire is fixed. Then the charge subsystem is described by the Hamiltonian (2) with the boundary condition (10). It is convenient to transform the variables $\phi_\rho(x) \to \phi_\rho(x) + \pi q(t)/\sqrt{2}$, i.e., to apply to the Hamiltonian a unitary transformation

$$U = \exp\left(-i\frac{\pi q(t)}{\sqrt{2}}\int_{-\infty}^{\infty}\Pi_\rho(x)\,dx\right). \tag{11}$$

Upon this transformation the boundary condition becomes time independent, $\phi_\rho(0) = 0$, and the Hamiltonian transforms into $\tilde{H}_\rho = U^\dagger H_\rho U - i\hbar U^\dagger\partial_t U$. The second term is a time-dependent perturbation that excites plasmons of very low frequency ω. The wavelength of these plasmons is large, $u_\rho/\omega \gg L$, and thus one can replace u_ρ and K_ρ with their values at $x \to \infty$; in particular, $K_\rho(\infty) = 1$, as $n_\infty a_B \gg 1$. The transformed Hamiltonian $\tilde{H}_\rho$ is conveniently presented in terms of the operators

$$b_k = \int_{-\infty}^{\infty}\theta(kx)\sin kx\left(\frac{\sqrt{|k|}}{\pi}\phi_\rho(x) + \frac{i}{\sqrt{|k|}}\Pi_\rho(x)\right)dx,$$

destroying plasmons with frequency $\omega_k = u_\rho(\infty)|k|$. Then $\tilde{H}_\rho$ takes the form

$$\tilde{H}_\rho = \hbar\int_{-\infty}^{\infty}\left(\omega_k b_k^\dagger b_k + i\frac{I(t)}{e}\,\frac{b_k - b_k^\dagger}{\sqrt{2|k|}}\right)dk. \tag{12}$$

At low frequency $\omega \ll T$ the time-dependent perturbation leads to both emission and absorption of plasmons with $k = \pm\omega/u_\rho(\infty)$. The energy dissipated into plasmon excitations in unit time is easily found by means of the Fermi golden rule, resulting in $W = \frac{1}{2}I_0^2(\pi\hbar/e^2)$. Thus, we conclude that $R_\rho = h/2e^2$.

To find R_σ one has to perform a similar calculation with a more complicated Hamiltonian (9). Performing the Jordan-Wigner transformation (7), we rewrite it as

$$H_\sigma = \frac{1}{2}\sum_l J[l + q(t)]\Big[(a_l^\dagger a_{l+1} + a_{l+1}^\dagger a_l) + 2\left(a_l^\dagger a_l + \frac{1}{2}\right)\left(a_{l+1}^\dagger a_{l+1} + \frac{1}{2}\right)\Big]. \tag{13}$$

In the absence of the external magnetic field the average spin per site $\langle S_l^z\rangle = 0$. In terms of the Hamiltonian (13) it means that the fermionic band is half-filled, $\langle a_l^\dagger a_l\rangle = \frac{1}{2}$. The exchange $J[l]$ takes its lowest value J at the center of the wire and grows up to $J[\infty] \sim E_F$ in the leads.

The Hamiltonian (13) can be easily treated if one neglects the interaction term in its second line. (This corresponds to the XY model of a spin chain, in which the z component of exchange vanishes.) Then the Hamiltonian describes a tight-binding model of noninteracting fermions with the bandwidth $2J[l]$ varying gradually between the small value $2J$ in the wire and a large value $\sim E_F$ in the leads. The fermions with energies $|\epsilon| < J$ pass through the constriction without backscattering, whereas the ones with $|\epsilon| > J$ are reflected. The time dependence $J[l + q(t)]$ can be interpreted as a slow movement of the constriction with respect to the Fermi gas, and leads to the dissipation of energy. For noninteracting fermions the calculation of W and the respective resistance is rather straightforward, and we find

$$R_\sigma^{XY} = \frac{h}{2e^2}\,\frac{1}{e^{J/T} + 1}. \tag{14}$$

At $T \ll J$ this result is exponentially small, $R_\sigma^{XY} \propto e^{-J/T}$, because at low temperature most of the quasiparticles pass the constriction without scattering. Only an exponentially small fraction of the excitations are scattered at the constriction and acquire the energy from the driving field. The resistance saturates at $T/J \to \infty$, when all the quasiparticles are backscattered.

The result (14) remains qualitatively correct for the full model (13). In particular, at $J \ll T$ the transport of spin fermions through the constriction is still suppressed, and R_σ saturates. In order to find R_σ at $T/J \to \infty$, one can notice that the bosonized model (3) is still applicable in the leads, where $J \sim E_F \gg T$. Then the absence of spin transfer through the constriction can be expressed as a hard-wall boundary condition upon ϕ_σ.

At $q(t) = 0$ this boundary condition may be presented without loss of generality as $\phi_\sigma(0) = 0$. At nonzero $q(t)$ the same condition $\phi_\sigma = 0$ is imposed at point $l = -q(t)$. The half-filling condition for the Hamiltonian (13) means that $q(t)/2$ fermions have passed through the wire at moment t. In analogy with the calculation of R_ρ, the boundary condition can be interpreted as an application of a fixed current $\dot{q}(t)/2$ of the spin fermions. The bosonization procedure leading from the Hamiltonian (13) to Eq. (3) expresses the current of

spin fermions as $\partial_t \phi_\sigma/\pi\sqrt{2}$. Thus, the appropriate boundary condition for the bosonized spin Hamiltonian (3) is $(\sqrt{2}/\pi)\phi_\sigma(0,t) = q(t)$. Note that this boundary condition is equivalent to Eq. (10), and the Hamiltonians (2) and (3) coincide in the leads, where $K_\rho = K_\sigma = 1$ and $g_{1\perp} = 0$. Therefore, one can complete the evaluation of R_σ by repeating the above calculation of R_ρ, and we conclude that $R_\sigma = h/2e^2$. The conductance of the wire $(R_\rho + R_\sigma)^{-1}$ reduces to e^2/h.

It is worth mentioning that the same boundary condition for the Hamiltonian (3) appears even at $T \lesssim J$ if a sufficiently large magnetic field B is applied. Indeed, if the Zeeman splitting is large compared to both T and the coupling J, the electrons in the wire are completely spin polarized, $\langle S_l^z \rangle = -\frac{1}{2}$. In terms of the Hamiltonian (13) this is interpreted as lowering the chemical potential below the bottom of the fermionic band in the wire, so that $\langle a_l^\dagger a_l \rangle = 0$. Thus the two leads are now separated by a barrier created by the narrowing fermionic band in the wire. The barrier is centered at $l = -q(t)$, and, upon bosonization, imposes the same boundary condition upon ϕ_σ as in the case of $B = 0$ and $T > J$. We therefore conclude that in a polarizing magnetic field $R_\sigma = h/2e^2$, and, as expected, the total conductance $G = e^2/h$.

At low temperature $T \ll J$ the model (13) to first approximation can be bosonized to the form (3) with position-dependent parameters u_σ, K_σ, and $g_{1\perp}$. At $T \to 0$ we have $K_\sigma = 1$ and $g_{1\perp} = 0$; i.e., the Hamiltonian (3) can be viewed as the bosonized version of the XY model. Then we conclude from Eq. (14) that at zero temperature $R_\sigma = 0$, and the conductance of the wire remains $2e^2/h$.

At finite length of the wire L the parameters of the Hamiltonian (3) do not reach their limiting values $K_\sigma = 1$, $g_{1\perp} = 0$. However, the resulting corrections to R_σ remain small in $1/nL$ even at $T \sim J$. A more interesting correction appears due to the fact that the bandwidth of the Hamiltonian (13) is finite, which is not accounted for accurately by the bosonization procedure leading to Eq. (3). If the wire is long, $nL \gg 1$, the spin chain (9) is nearly uniform at each point l. Its low energy excitations are conveniently classified [10] in terms of spinon quasiparticles with spectrum $\epsilon = \frac{\pi}{2} J[l] \sin p$, where p is the wave vector in the lattice model. At low temperature $T \ll J$ typical excitations have energies $\epsilon \sim T$ small compared to the bandwidth in the center of the constriction. These spinons pass through the wire without scattering and do not interact with the driving field. On the other hand, a small fraction of the spinons have energies exceeding $\pi J/2$. These excitations are not supported by the spin chain at the center of the wire, and thus are reflected by the constriction. Similar to the case of the XY model, the reflected particles acquire energy from the driving field and contribute to the dissipation. The density of such spinons is exponentially small, and we get

$$R_\sigma \sim R_0 \exp\left(-\frac{\pi J}{2T}\right). \tag{15}$$

This treatment does not enable one to evaluate R_0. The analogy with the XY model suggests $R_0 \sim h/e^2$.

Experimentally, corrections to the quantized conductance showing activated temperature dependence consistent with (15) have been reported in short wires [12]. At higher temperatures the conductance tends to saturate at $0.7 \times (2e^2/h)$ instead of e^2/h. Quantization of conductance at e^2/h in the absence of magnetic field was reported in longer wires at low electron density [13].

To summarize, in the regime of strong interactions, $na_B \ll 1$, the electrons in a quantum wire form a Wigner crystal with exponentially small antiferromagnetic spin exchange J. At $T \ll J$ the conductance remains $2e^2/h$ up to exponentially small corrections, Eq. (15). At $J \ll T$ the conductance drops to e^2/h. Remarkably, this result does not assume spontaneous spin polarization in the wire [14].

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