Maximizing the Hilbert Space for a Finite Number of Distinguishable Quantum States

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Given a particular quantum computing architecture, how might one optimize its resources to maximize its computing power? We consider quantum computers with a number of distinguishable quantum states, and entangled particles shared between those states. Hilbert-space dimensionality is linked to nonclassicality and, hence, quantum computing power. We find that qutrit-based quantum computers optimize the Hilbert-space dimensionality and so are expected to be more powerful than other qudit implementations. In going beyond qudits, we identify structures with much higher Hilbert-space dimensionalities.

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Quantum computation [1] has, in a remarkably short time, become one of the most interesting fields in applied quantum physics today. There are numerous suggestions for implementing quantum computers. To construct a practical quantum computer (QC) with any implementation is clearly a highly demanding task. It is therefore essential that, for a given physical architecture, one optimize the available resources to give the most powerful computer possible. The Hilbert-space dimension of a QC has been identified as its primary resource [2]. In this Letter, we direct attention to the important issue of maximizing the Hilbert-space dimension for a given QC architecture.

Most of the QC literature focuses on qubits (quantum bits) as the fundamental quantum elements of computing. There is a certain attraction in working with qubits: They are the simplest systems in which to observe quantum effects; we have much experience manipulating two-state systems, and some classical concepts based on bits map well to qubits. However, a QC is not limited to qubits, and we will show that more general structures can maximize the Hilbert-space dimension.

One common feature of all scalable quantum computers is their use of entangled particles [2]. If we consider a QC comprising a fixed number of entangled particles, obviously the total Hilbert-space dimensionality will be increased by increasing the internal degrees of freedom (e.g., spin) of each particle. However, one cannot usually change the dimensionality of the particle state space; nor would the added complexity involved be necessarily favorable. On the other hand, there are many QC architectures, involving quantum particles having access to a finite number of distinguishable quantum states. Examples include charge-qubit schemes [3,4], superconducting Cooper-pair boxes [5], quantum computing based on photons in interferometers joined by lossless nonlinear optical elements [6], and the linear optics implementation [7]. All these proposals have an architecture (e.g., a quantum dot array) which defines the quantum states with the particles in some subset of those states. In this work, we consider what grouping of particles and states used for quantum information processing maximizes the total Hilbert-space dimensionality. We term these groups *quantum elements*.

Qudits (quantum digits) [8] are quantum elements that generalize qubits. In a qudit, the number of states is allowed to be any integer greater than unity. The qubit is therefore the two-state qudit, and the qutrit is the three-state qudit. Investigations into qudits have shown many significant results. Entanglement between two qutrits was first discussed by Caves and Milburn [9]. Bell inequalities for systems of qudits are more strongly violated than analogous systems of qubits [10], and recent experiments have shown entanglement of two photonic qutrits, realized using the orbital angular momentum as the quantum state [11]. A quantum-communication protocol using qutrits has also been proposed [12]. Universal quantum gates and gate fidelity have been studied for qudit systems [13], a readout method based on quantum state tomography has been proposed [14], and the extension of infinite-dimensional qudits to continuous-variable quantum computation has also been made [15].

The simplest architecture one can consider is a qudit based QC, i.e., systems with one particle per quantum element. We refer to the x-state qudit as an x qudit. We also consider a more general case, with a total of N quantum sites, x sites per quantum element, each quantum element having k particles, and no more than l particles at each site. We call these quantum elements (k, l) packing x qudits and use the shorthand notation $Q_x(k, l)$ to describe them. As we are concerned here with dimensionalities, we denote the dimensionality of a $Q_x(k, l)$ quantum element by $d_x(k, l)$. To illustrate our notation, x qudits would be written $Q_x(1, 1)$; Blume-Kohout *et al.* [2] described the dimensionality of fermionic and bosonic systems, which in our notation would be $Q_x(k, 1)$ and $Q_x(k, k)$, respectively; Creffield *et al.* performed analysis of two electrons in a polygonal dot [16], which

corresponds to $Q_x(2, 1)$; Creffield and Platero considered two electrons in a square dot with the possibility of double occupancy [17] [$Q_4(2, 2)$], and without double occupancy this system has been proposed as a scalable quantum element by Jefferson *et al.* [18] [$Q_4(2, 1)$].

In view of the above generalizations, it is useful to consider whether there might be some quantum algorithms which might be more appropriately solved on such a machine. Although we are not able to adequately address this question, we do note that Abrams and Lloyd [19] discovered an efficient quantum algorithm for simulation of many-body Fermi systems. It is therefore sensible to propose the reverse of this method, namely, that these more general structures are able to efficiently perform quantum algorithms; this thesis is in keeping with Bravyi and Kitaev's work on fermionic quantum computation [20].

Despite the general nature of our concepts, it is useful to keep a concrete QC proposal in mind when considering these schemes. We concentrate on a charge scheme related to that of Hollenberg *et al.* [4]. In this scheme, a finite number of ionized donors (our quantum sites) is introduced into a solid matrix (phosphorus in silicon), with top gates to control the barrier potentials, and, hence, tunneling rates, between the sites, and the relative energy levels on each site. We extend the original proposal (which was for charge qubits only) to include qudits and packing structures. A more detailed study of a charge qudit QC appears in Ref. [21].

Our model for understanding Hilbert-space optimization is a linear array of N donor impurities, with controllable tunneling probabilities and individual energy levels. If we partition the space into x qudits $[Q_x(1, 1)]$, shown in Fig. 1 for $x = 2, 3, 4, 6, 12$ and $N = 12$, then we have N/x quantum elements, and the dimension of the Hilbert space obtained by maximally entangling these N/x electrons is

$$[d_x(1, 1)]^{N/x} = x^{N/x}. \tag{1}$$

By differentiating Eq. (1), we immediately see that the total dimensionality is maximized for $x = e$. As the number of quantum sites per particle must be an integer, we note that for identical qudits the dimensionality is optimized for qutrits. The Hilbert-space dimensionality for qutrits over qubits will be larger by a factor $\mathcal{D}_3/\mathcal{D}_2 = \exp[N(\ln 3/3 - \ln 2/2)]$. This simple result carries an important message about the merits of nonqubit QCs, and adds further motivation to experimental drives to realize qudits.

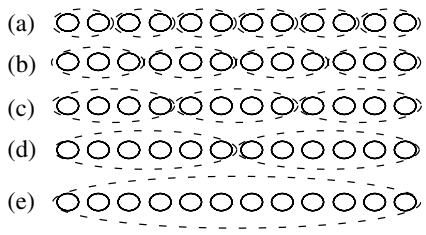

FIG. 1. Possible groupings of a linear array of 12 distinguishable quantum sites into various quantum qudit partitions. Qudit groupings are shown schematically as dashed ovals, and open circles are quantum sites. (a) Partitioning into six qubits, (b) partitioning into four qutrits, (c) partitioning into three 4-qudits, (d) partitioning into two 6-qudits, (e) partitioning into one 12-qudit.

In Fig. 2, we show the Hilbert-space dimensionality of a twelve-state system as a function of the size of each quantum element. Each different case is explained below. The + symbols in Fig. 2 show the dimensionality associated with qudit arrangements. As expected, the dimensionality is maximized by a qudit size of 3 (we show only integer qudit sizes). For the system with 12 quantum states, $[d_2(1, 1)]^6 = 64$, whereas $[d_3(1, 1)]^4 = 81$.

Conceivably, it may be possible to realize a quantum computer using a mixture of qubits and qutrits, as was recently discussed by Daboul *et al.* [22], to optimize the total dimensionality of the Hilbert space. Suppose we partition the space into y_2 qubits and y_3 qutrits such that $2y_2 + 3y_3 = N'$, where $N' \leq N$. One might guess that the dimensionality of the Hilbert space would be increased by mixing qubits and qutrits such that the average dimensionality of the system $(2y_2 + 3y_3)/(y_2 + y_3)$ is closer to the optimal value e. However, this is not the case. The dimensionality of the Hilbert space of a system partitioned as suggested is given by $\mathcal{D} = 2^{y_2}3^{y_3} = 2^{(N'-3y_3)/2}3^{y_3}$. $d_e(1, 1) = \exp(N'/e)$, which leads to $\mathcal{D}/d_e(1, 1) = \exp(my_3 + b)$, where $m = \ln(3) - \frac{3}{2}\ln(2) > 0$ and $b = [\frac{\ln(2)}{2} - e^{-1}]N'$. Note that $\exp(my_3 + b) = 1$ if $y_3 = -b/m \approx 0.36N'$. Since $y_3 \leq N'/3$ on physical grounds, this implies $my_3 + b < 0$; i.e., the exponential is always less than one. Furthermore, since $m > 0$, the value of the exponential *increases* with the number of qutrits y_3. Thus, the dimensionality of the Hilbert space is optimized by

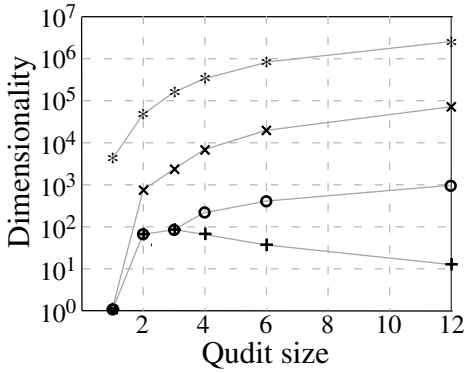

FIG. 2. Hilbert space dimensionality for various geometries as a function of qudit size for N distinguishable quantum states. + corresponds to equipartitioning into qudits, ○ to a $Q_x(x/2, 1)$ partition, × to a $Q_x(x, 2)$ partition, and * to spin referenced systems. Note that the dimensionalities are defined only when N/x is a positive integer, i.e., $x = 2, 3, 4, 6, 12$. The lines are merely guides for the eye.

partitioning the system into the largest available number of qutrits, except where adding a single qubit would more efficiently use all of the available quantum states.

We now consider the case where the number of particles per quantum element is allowed to vary, but there can be no more than one particle per quantum state, i.e., the $Q_x(k, 1)$ partition, or fermionic system. To maximize the total Hilbert-space dimensionality, the number of particles per quantum element is given by $x = 2k$ (x even) or $x = 2k + 1$ (x odd). The total dimensionality for a system with N sites, partitioned into N/x elements is (for x even and N/x an integer)

$$[d_x(x/2, 1)]^{N/x} = \binom{x}{x/2}^{N/x},$$

where $\binom{n}{m}$ is the number of possible identifiably different permutations of n elements of two different types, m of which are of type 1 (e.g., particles) and $n - m$ of which are of type 2 (e.g., holes or empty sites). The Hilbert-space dimensionality showing this form of partitioning is represented by the open circles, ○, in Fig. 2. The total dimensionality, $[d_x(x/2, 1)]^{N/x}$, is clearly maximized when $x = N$, which corresponds to all particles roaming freely on the lattice. This is the case of local fermionic modes discussed in Ref. [20], and is scalable, according to Ref. [2]. In a 12-state configuration, for example, Fig. 3(e) shows the configuration which maximizes the dimensionality of the Hilbert space, with $d_{12}(6, 1) = 924$, an order of magnitude larger than for qutrits. Despite this maximization, it is important to realize that there may be advantages to partitioning into smaller subspaces (i.e., $x < N$). For example, it is not possible to do error correction without some redundant elements.

One natural extension is to allow the maximum number of particles per site to increase to two. This could physically correspond to an electron-based QC, where the charging energy of two electrons on one dot can be overcome [17]. In this case, the dimensionality is

$$[d_x(k, 2)]^{N/x} = \left[\sum_{j_2=0}^{k/2} \binom{x - j_2}{k - 2j_2}\binom{x}{j_2}\right]^{N/x},$$

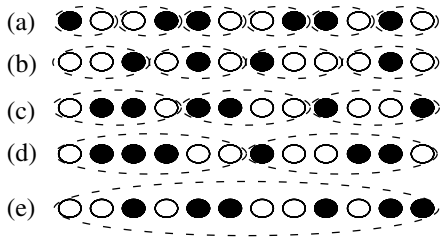

FIG. 3. Possible groupings of a linear array of 12 distinguishable quantum sites into various $Q_x(k, 1)$ partitions. The x-qudit groupings are shown schematically as dashed ovals, open circles are quantum sites, and filled circles correspond to (randomly chosen) filled sites. (a) Six $Q_2(1, 1)$ partitions, (b) four $Q_3(1, 1)$ partitions, (c) three $Q_4(2, 1)$ partitions, (d) two $Q_6(3, 1)$ partitions, and (e) one $Q_{12}(6, 1)$ partition.

where j_2 indexes the number of sites with two electrons per site and is allowed to range from 0 to $N/2$ for x even, or 0 to $(N - 1)/2$ for x odd. One can show that the dimensionality of each quantum element is maximized when $k = x$; i.e., the number of particles per element is equal to the number of sites per element. The dimensionality in this case is shown by the crosses, ×, in Fig. 2, which is significantly larger than the case where at most one particle is allowed per site.

Another alternative is where we have an extra quantum number (for example spin), which would double the number of accessible quantum states. In this case, the system returns to the standard fermionic case [maximized by $Q_{2x}(x, 1)$] and the total dimensionality becomes simply

$$[d_{2x}(x, 1)]^{N/x} = \binom{2x}{x}^{N/x}.$$

This dimensionality is shown by the asterisks,*, in Fig. 2 which represents the largest dimensionality of all the cases under consideration.

If we generalize to l particles per quantum state, the general result for the Hilbert-space dimension of a single quantum element is

$$d_x(k, l) = \sum_{[j_1 \cdots j_l]} \prod_{s=1}^{l} \binom{x - \sum_{r=s+1}^{l} j_r}{j_s},$$

where $j_r \in \{0, \ldots, l\}$ denotes the number of sites with r particles, and the sum is over all sets $[j_1 \cdots j_l]$ of integers that satisfy $\sum_{r=1}^{l} r j_r = k$ and $\sum_{r=1}^{l} j_r \leq x$.

With no restriction on the number of particles per site, we retrieve the bosonic case. We would then be extending familiar bosonic QC schemes to the multiple-particle limit, for example, linear optical [7], nonlinear optical [6], or superconducting [5] QC schemes. The dimensionality is considerably simplified and is [2,23]

$$d_x(k, k) = \binom{x + k - 1}{x - 1}.$$

This dimensionality can greatly exceed the fermionic case, for the same number of accessible quantum states, for a large enough number of particles. Furthermore, this dimensionality monotonically increases with increasing particle number. By judicious choice of architecture, it may be easier to improve quantum computing power in a bosonic QC after initial construction (i.e., to upgrade) than for other architectures.

There is certainly far more to quantum computers than optimizing the Hilbert-space dimension. Here we have not entered into issues of operational complexity, which will be specific to particular architectures, as well as decoherence which is expected to differ substantially between implementations. However, comparing qudit implementations, a system comprising qubits alone will maximize the amount of information that can be obtained in a single measurement step. This is because an

optimum measurement would be to measure the state of every quantum particle. For qubits, that would yield $N/2$ values; for qutrits, it would yield only $N/3$ values. As a rough measure, we may say that, to equalize the amount of information gained about the state of the computer, we would need to perform $(N/2)/(N/3) = 1.5$ times as many measurements on a qutrit-based quantum computer as a qubit-based quantum computer. The dimensionality of the qutrit Hilbert space exceeds that of the qubit Hilbert space by 1.5 when

$$\exp\left[N\left(\frac{\ln 3}{3} - \frac{\ln 2}{2}\right)\right] > 1.5, \qquad \Rightarrow N > 20.65. \quad (2)$$

Therefore when the number of quantum sites exceeds 20, the increase in dimensionality of the Hilbert space should compensate for the increased measurement complexity.

We have shown, using simple mathematical arguments, that for a quantum computer with a finite number of distinguishable states to be shared between a finite number of quantum particles arranged in qudits the Hilbert space is maximized when the system is partitioned into the largest possible number of qutrits. Given that the construction of quantum computers is highly demanding, this kind of optimization for a given geometry may make the difference between a practical and impractical implementation of quantum computing. This is an added constraint over and above scalability. The Hilbert-space dimensionality can be increased substantially by both relaxing the requirement that the particles be partitioned into qudits and by implementing what we have termed a packing geometry where the particles are allowed to be present in any site, up to some maximum, determined by the physical properties of the quantum system.

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