Universality of Entropy Scaling in One Dimensional Gapless Models

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We consider critical models in one dimension. We study the ground state in the thermodynamic limit (infinite lattice). We are interested in an entropy of a subsystem. We calculate the entropy of a part of the ground state from a space interval $(0, x)$. At zero temperature it describes the entanglement of the part of the ground state from this interval with the rest of the ground state. We obtain an explicit formula for the entropy of the subsystem at any temperature. At zero temperature our formula reproduces a logarithmic formula, discovered by Vidal, Latorre, Rico, and Kitaev for spin chains. We prove our formula by means of conformal field theory and the second law of thermodynamics. Our formula is universal. We illustrate it for a Bose gas with a delta interaction and for the Hubbard model.

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Conformal field theory [1,2] is useful for the description of the low temperature behavior of gapless models in one space and one time dimensions. Critical models are classified by a central charge c of the corresponding Virasoro algebra. A formal definition of the central charge is given the Appendix. Roughly speaking, the central charge counts the number of gapless degrees of freedom. Conformal field theory is closely related to the Luttinger liquid [3] approach. We are interested in the specific entropy s (entropy per unit length). Let us start with a specific heat $C = T ds/dT$. This low temperature behavior was obtained in [4,5]:

$$C = \frac{\pi T c k_B^2}{3\hbar v}.$$

Here c is a central charge and v is Fermi velocity. We put both Planck and Boltzmann constants equal to $k_B = \hbar = 1$. Later in the paper we shall use

$$C = \frac{\pi T c}{3v} \quad \text{as} \quad T \to 0. \tag{1}$$

We are more interested in the specific entropy s. We can integrate the equation and fix the integration constant from the third law of thermodynamics ($s = 0$ at $T = 0$). For the specific entropy we have the same low temperature behavior:

$$s = \frac{\pi T c}{3v} \quad \text{as} \quad T \to 0. \tag{2}$$

For quantum spin chains, this formula agrees with [6]. To formulate the problem precisely, let us consider a Bose gas with delta interaction. The Hamiltonian of the model is

$$H = \int dx(\partial\psi_x^\dagger \partial\psi_x + g\psi^\dagger\psi^\dagger\psi\psi). \tag{3}$$

Here ψ is a canonical Bose field and $g > 0$ is a coupling constant. The model was solved in [7]. The physics of the model is described in a book [8]. First, let us consider the model at zero temperature and in infinite volume. The ground state is unique $|gs\rangle$. We consider the positive density case. We are interested in the entropy $S(x)$ of the part of the gas present on the space interval $(0, x)$. Formally, we can define it by means of the density matrix

$$\rho = \mathrm{tr}_\infty(|gs\rangle\langle gs|). \tag{4}$$

Here we traced out the "external" degrees of freedom; they describe the gas on the rest of the ground state: on the unification of the intervals $(-\infty, 0)$ and (x, ∞). The density matrix ρ describes gas on the interval $(0, x)$. Now we can calculate von Neumann entropy $S(x)$ of the part of the gas on the interval $(0, x)$:

$$S(x) = -\mathrm{tr}_x \rho \ln\rho. \tag{5}$$

Here we are taking the trace with respect to the degrees of freedom representing the part of the gas on the interval $(0, x)$. In the major textbooks, it is shown that the laws of thermodynamics can be derived from statistical mechanics (see, for example, [9,10]). The second law of thermodynamics states that the entropy is extensive parameter: the entropy of a subsystem $S(x)$ is proportional to the system size x:

$$S(x) = sx \qquad \text{at } T > 0. \tag{6}$$

Thermodynamics is applicable to the subsystem of macroscopical size, meaning large x. Here the specific entropy s depends on the temperature. For small temperatures the dependence simplifies [see (2)]:

$$S(x) = \frac{\pi T c}{3v} x, \qquad x > \frac{1}{T}. \tag{7}$$

Let us try to find out how $S(x)$ depends on x for zero temperature. It is some function of the size

$$S(x) = f(x) \quad \text{at} \quad T = 0. \tag{8}$$

Now let us apply the ideas of conformal field theory (see [2,4,5] and also Chap. XVIII of [8]). We can arrive at small temperatures from zero temperature by conformal mapping $\exp(2\pi T z/v)$. It maps the whole complex plane of z without the origin to a strip of the width $1/T$. This replaces zero temperature by positive temperature T. The conformal mapping results in a replacement of variable x by $(v/\pi T)\sinh(\pi T x/v)$. The entropy of the subsystem at temperature T is given by the formula

$$S(x) = f\left(\frac{v}{\pi T}\sinh\left[\frac{\pi T x}{v}\right]\right) \quad \text{at} \quad T > 0. \tag{9}$$

In order to match this to formula (7), we have to consider an asymptotic of large x. The formula simplifies:

$$S(x) = f\left(\exp\left(\frac{\pi T(x - x_0)}{v}\right)\right), \qquad Tx \to \infty. \tag{10}$$

Here $\pi T x_0/v = -\ln(v/2\pi T)$.

Formulas (7) and (10) should coincide. Both represent the entropy of the subsystem for small positive temperatures. This provides an equation for f:

$$f\left(\exp\left(\frac{\pi T(x - x_0)}{v}\right)\right) = \frac{\pi T c}{3v}(x - x_0). \tag{11}$$

This formula describes the asymptotic for large x, so we added $-x_0$ to the right-hand side. We are considering the region $x > 1/T$ and $x_0 \sim \ln(1/T)$, so $x \gg x_0$ at $T \to 0$. In order to solve the equation for f, let us introduce a new variable $y = \exp[\pi T(x - x_0)/v]$. Then Eq. (11) reads

$$f(y) = \frac{c}{3}\ln y. \tag{12}$$

We found the function f in (8). Now we know that at zero temperature the entropy of the gas containing on the interval $(0, x)$ is

$$S(x) = \frac{c}{3}\ln x \quad \text{as} \quad x \to \infty. \tag{13}$$

Let us recall that for a Bose gas the central charge $c = 1$ (see [8]). Our result agrees with the third law of thermodynamics. The specific entropy is a limit of the ratio $S(x)/x$ as $x \to \infty$. The limit is zero.

Now we can go back to our formula (9) and substitute the expression for f, which we found:

$$S(x) = \frac{c}{3}\ln\left(\frac{v}{\pi T}\sinh\left[\frac{\pi T x}{v}\right]\right). \tag{14}$$

This formula describes crossover between zero and large temperature.

The proof presented here is universal. It is also applicable to quantum spin chains; for example, to the XXZ spin chain. The Hamiltonian of the model is

$$\mathbf{H} = -\sum_j \{\sigma_j^x\sigma_{j+1}^x + \sigma_j^y\sigma_{j+1}^y + \Delta\sigma_j^z\sigma_{j+1}^z + 2h\sigma_j^z\}. \tag{15}$$

It describes the interaction of spins 1/2. Here σ are Pauli matrices. The model can be described by a conformal field theory with the central charge $c = 1$ in a disordered regime: $-1 \le \Delta < 1$ and magnetic field h is smaller than critical h_c (see [8]). For a magnetic field larger than critical, the ground state is ferromagnetic (all spins are parallel). The ferromagnetic state does not have any entropy or entanglement.

For the spin chains, the formula (13) was discovered in [1,11]. Vidal, Latorre, Rico, and Kitaev emphasized the role of the entropy of a subsystem in quantum computing. The entropy of the subsystem $S(x)$ describes the entanglement of the subsystem and the rest of the ground state.

The formula (13) describes the entropy of a large subsystem $x \to \infty$ at zero temperature; the first correction to this formula is a constant term:

$$S(x) = \frac{c}{3}\ln\frac{x}{x_0} \quad \text{as} \quad x \to \infty.$$

The constant term defines the scale x_0. It does not depend on x, but it does depend on the magnetic field and anisotropy. Reference [12] showed that the magnetic field increases x_0 and reduces the entanglement.

Thus far, we have discussed the entropy of entanglement for critical models. For noncritical (gap-full) models, we expect that the asymptotic of $S(x)$ will be a constant, according to Ref. [1].

Hubbard model.—The Hamiltonian for the Hubbard model H can be represented as

$$H = -\sum_{\substack{j=1\\ \sigma=\uparrow\downarrow}}(c_{j,\sigma}^\dagger c_{j+1,\sigma} + c_{j+1,\sigma}^\dagger c_{j,\sigma}) + u\sum_{j=1} n_{j,\uparrow}n_{j,\downarrow} - h\sum_{j=1}(n_{j,\uparrow} - n_{j,\downarrow}). \tag{16}$$

Here $c_{j,\sigma}^\dagger$ is a canonical Fermi operator on the lattice (operator of creation of an electron) and $n_{j,\sigma} = c_{j,\sigma}^\dagger c_{j,\sigma}$ is an operator on the number of electrons in cite number j with spin σ. Summation in the Hamiltonian goes through the whole infinite lattice. The coupling constant $u > 0$, and the magnetic field h is below critical. We are considering the model below half filling (less than one electron per lattice cite). The model was solved in [13]. A collection of important papers on the model can be found in [14]. Charge and spin are separate in the model. The model is gapless. Both charge and spin degrees of freedom can be described by Virasoro algebra with central charge equal to 1 (see [15]):

$$c_c = 1 \quad \text{and} \quad c_s = 1. \tag{17}$$

Also, Fermi velocities are different for spin v_s and charge v_c degrees of freedom. Both spin and charge degrees of freedom contribute to the specific entropy s. For small temperatures, we have

$$s = \frac{\pi T}{3v_s} + \frac{\pi T}{3v_c}. \tag{18}$$

For small positive temperatures, the entropy of the subsystem is proportional to the size of the system (second law):

$$S(x) = \left(\frac{\pi T}{3v_s} + \frac{\pi T}{3v_c}\right)x. \tag{19}$$

Now we have to apply conformal arguments separately to spin and charge degrees of freedom. The results add up:

$$S(x) = \frac{2}{3}\ln x. \tag{20}$$

This describes the entropy of electrons on the interval $(0, x)$ in the infinite ground state at zero temperature. This is actually an asymptotic for large x. In the case of the Hubbard model, the crossover formula (which mediates between zero and large temperature) looks like this:

$$S(x) = \frac{1}{3}\ln\left[\frac{v_s}{\pi T}\sinh\left(\frac{\pi T x}{v_s}\right)\right] + \frac{1}{3}\ln\left[\frac{v_c}{\pi T}\sinh\left(\frac{\pi T x}{v_c}\right)\right]. \tag{21}$$

The approach developed here is universal. It is applicable to other models of strongly correlated electrons (see [14]). For example, formulas (20) and (21) describe entropy in the t-J model as well.

In this Letter, we described the universal properties of the entropy in one dimensional gapless models. We studied the entropy of the subsystem. We considered scaling of the entropy in spin chains, the Hubbard model, and a Bose gas with delta interaction. We discovered a crossover formula for the entropy of the subsystem (see (14) and (21). It describes the entropy of a large subsystem at any temperature; it mediates between a logarithmic function for zero temperature and a linear function for large temperature. In the case of zero temperature, we proved the logarithmic formula (13) discovered in [1]. We think that the renormalization group approach developed in [16–22] will be useful for further analysis of entanglement. It is based on perturbed conformal field theory.

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Appendix.—An important characteristic of conformal field theory is central charge. It can be defined by considering an energy-momentum tensor $T_{\mu,\nu}(z)$. Here z is a complex space-time variable $z = x + ivt$ and v is a Fermi velocity. The component of energy-momentum tensor with $\mu = \nu = z$ is denoted by $T_{z,z} = T$. The correlation function of this operator has a singularity:

$$\langle T(z)T(0)\rangle = \frac{c}{z^4}.$$

The coefficient c is the central charge.

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[1] G. Vidal, J. I. Latorre, E. Rico, and A. Kitaev, Phys. Rev. Lett. **90**, 227902 (2003); quant-ph/0211074.
[2] A. A. Belavin, A. M. Polyakov, and A. B. Zamolodchikov, Nucl. Phys. **B241**, 333 (1984).
[3] F. D. M. Haldane, Phys. Rev. Lett.,**47**, 1840 (1981).
[4] I. Affleck, Phys. Rev. Lett. **56**, 746 (1986).
[5] H.W. Blote, J. L. Cardy, and M. P. Nightingale, Phys. Rev. Lett. **56**, 742 (1986).
[6] M. Takahashi, *Thermodynamics of One-Dimensional Solvable Models* (Cambridge University Press, Cambridge, England, 1999).
[7] E. H. Lieb and W. Liniger, Phys. Rev. **130**, 1605, (1963).
[8] V. E. Korepin, N. M. Bogoliubov, and A. G. Izergin, *Quantum Inverse Scattering Method and Correlation Functions* (Cambridge University Press, Cambridge, England, 1993).
[9] K. Huang, *Statistical Mechanics* (Wiley, New York, 1987).
[10] L. D. Landau *et al.*, *Statistical Physics* (Butterworth-Heinemann, London, 2002), 3rd ed.
[11] C. Holzhey, F. Larsen, and F. Wilczek, Nucl. Phys. **B424**, 443 (1994).
[12] B.-Q. Jin and V. E. Korepin, quant-ph/0304108.
[13] E. H. Lieb and F. Y. Wu, Phys. Rev. **20**, 1445 (1968).
[14] F. H. L. Essler and V. E. Korepin, *Exactly Solvable Models of Strongly Correlated Electrons* (World Scientific, Singapore, 1994).
[15] H. Frahm and V. Korepin, Phys. Rev. B,**42**, 10 553 (1990).
[16] R. R. P. Singh, M. E. Fisher, and R. Shankar, Phys. Rev. B **39**, 2562 (1989).
[17] I. Affleck, Phys. Rev. Lett.**56**, 746 (1986).
[18] I. Affleck, D. Gepner, H. J. Schultz, and T. Ziman, J. Phys. A **22** , 511 (1989).
[19] I. Affleck, cond-mat/9802045.
[20] S. Lukyanov, cond-mat/9712314.
[21] S. Lukyanov, cond-mat/9809254.
[22] S. Lukyanov, cond-mat/9712314.