Stationary Rarefaction Wave in Magnetized Hall Plasmas

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We demonstrate the existence of a stationary rarefaction wave in a current-carrying plasma. The result unexpectedly mismatches with the commonly accepted viewpoint about the impossibility of rarefaction shocks in gases or plasmas. The discovered wave may appear when the magnetic field has penetrated into the plasma and magnetized the electrons. At this stage, the wave front is formed at the cathode and propagates towards the anode through the magnetized quasineutral plasma. The case of low collisionality is investigated analytically. This phenomenon could explain the recent surprising experimental observations of a local plasma density drop in several laboratory plasmas.

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During the last decade the so-called pulse-power plasma devices became sources of the highest electrical power ever achieved in the laboratory [1]. The classical problem of the interaction of a magnetic field with plasmas acquired particular interest for these physical objects (e.g., plasma filled diodes and switches or different types of Z pinches). Indeed, the magnetic field is usually known to penetrate a plasma conductor through diffusion. However, already the first space- and time-resolved measurements of the magnetic field evolution in low-density pulsed plasmas unexpectedly demonstrated a different, faster than diffusion, more shocklike fashion of the field flowing [2]. Almost at the same time [3], it was theoretically stated that the frozen-in law for a strong magnetic field indeed could lead to its fast penetration appearing as a solution of a Burgers-type equation. Provided an initial density gradient, magnetic field propagation occurs as a shock with the velocity

$$\mathbf{u}_{\rm KMC} = (c/8\pi e n_e)[\mathbf{B} \times \nabla \ln n_e] \qquad (1)$$

($\mathbf{B}$ is the magnetic field, n_e is the electron density, c is the speed of light, and e is the electron charge).

As usual, reasonable agreement between the first experiments and theory, as well as practical importance of the physical objects under interest provoked extensive theoretical study of the phenomenon (see the review in [4]). Today, the plasmas considered in these studies are already widely referred to as Hall plasmas. The most common feature here is that the electrons are magnetized, the ions are not, and the Hall electric field $\mathbf{E}_H = -[\mathbf{v}_e \times \mathbf{B}]/c$ must be kept in Ohm's law.

There exists another interesting experimental observation in this class of plasmas, which has not been reliably explained yet. Experiments [5–7] showed that the plasma density was abruptly decreasing by about 1 order of magnitude at the characteristic lengths comparable to the plasma size. Unexpectedly, instead of producing plasma compression, the magnetic field could first propagate on the background of almost motionless ions and could cause then substantial plasma rarefaction [7].

In explaining this unusual behavior of the magnetized plasmas, let us pursue the following logic. In the fast shocks of Eq. (1) ($u_{\rm KMC} \gg v_A$, v_A is the Alfvén velocity), the plasma compression is low because the ions are affected by E_H only during a short time [6,8]. On the contrary, behind the penetration front, this field may accelerate the ions until they exit the system creating conditions for local plasma rarefaction (see Fig. 1). This physical scenario was pointed out previously by Rudakov [9]. The present Letter grounds this possibility in the framework of the Hall MHD and theoretically discovers that this process acquires the form of a stationary shocklike rarefaction wave. It is interesting to note here that in conventional hydrodynamics the existence of rarefaction shocks is prohibited in media with normal thermodynamic properties (the so-called Zemplen theorem [10]).

We will consider the problem in plane geometry shown in Fig. 1, where a quasineutral plasma is already magnetized after the process of Eq. (1). In our frame of reference we have for the fields $\mathbf{B} \equiv (0, -B, 0)$, $B > 0$, $\mathbf{E} \equiv (E_x, 0, E_z)$, and $\mathbf{v}_i \equiv (V_x, 0, V_z)$, $\mathbf{v}_e \equiv (v_x, 0, v_z)$ for the ion and electron velocities, respectively. The characteristic sizes are such that $\Delta \ll h$ and $\partial/\partial z \sim 1/h \ll \partial/\partial x$. In addition, $|V_x| \gg |V_z|$ and $v_z \gg v_x$. We neglect the thermal pressure, $B^2 \gg nT$. The important point of our problem is that initially the magnetic field may represent a weak monotonic decreasing function of z, $B = B_0(z)$.

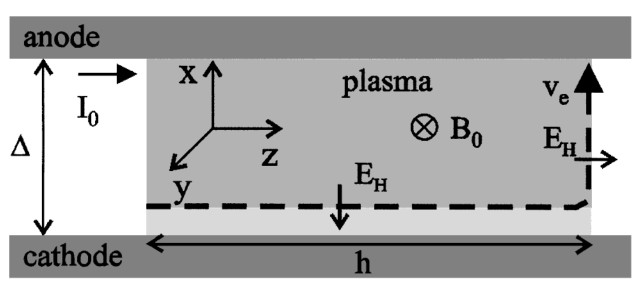

FIG. 1. The magnetic field B_0 is generated by an external current I_0. Anode and cathode may represent either electrodes or nonmagnetized, denser [see Eq. (1)] plasma regions.

This assumption is grounded, see Eq. (1), in the plasma part where the density decreases with z, for example, towards the downstream plasma boundary with vacuum. Therefore, we will normalize the magnetic field and plasma density to the maximal initial values in the nonperturbed plasma, B_m and n_m accordingly. After that, we normalize time to $\omega_{\rm ci}^{-1}$ (inversed ion cyclotron frequency), coordinates to $c/\omega_{\rm pi}$ ($\omega_{\rm pi}$ is the ion plasma frequency), velocities to v_A, electric field to the value $B_m v_A/c$, and collisional frequency to $\omega_{\rm ce}$ (electron cyclotron frequency). All the values are calculated upon B_m and n_m.

In this problem statement, we would obtain, for example, for the ion momentum equation:

$$n\frac{dV_x}{dt} \approx n\left(\frac{\partial V_x}{\partial t} + V_x\frac{\partial V_x}{\partial x}\right) = -b\frac{\partial b}{\partial x}, \qquad (2)$$

where, for ions or electrons, the total time derivative is as usual $d/dt = \partial/\partial t + (\mathbf{v}_{i,e}\nabla)$. Continuing to use our approximations, the Ampere's law and continuity equations will appear as

$$\frac{\partial b}{\partial z} = n(V_x - v_x), \qquad \frac{\partial b}{\partial x} = n v_z, \qquad (3)$$

$$\frac{\partial n}{\partial t} + \frac{\partial}{\partial x}(nV_x) = 0. \qquad (4)$$

Let us now suppose that there propagates a wave in the X direction with the front velocity $u(z) > 0$, so that $\partial/\partial t = -u\partial/\partial x$. Integration of Eqs. (2) and (4) across the front yields

$$\frac{1}{n} = \frac{1}{n_0} + \frac{b_0^2 - b^2}{2n_0^2u^2}, \qquad V_x = \frac{b^2 - b_0^2}{2n_0u}. \qquad (5)$$

In difference with a similar system of equations used in [8] we suppose $b = b_0(z)$ and $n = n_0(z)$ in initially magnetized nonperturbed plasma. Besides, we will seek for a solution with $u > 0$ and $V_x < 0$, when $n < n_0$ and $b < b_0$ behind the wave front. Equation (5) does not yet respond to the question about the character of the shock wave (compression or rarefaction). So, we proceed with our analysis using the generalized Ohm's law

$$0 = -E_x - v_z b, \qquad \varepsilon\frac{dv_z}{dt} = -E_z + v_x b - \nu v_z, \qquad (6)$$

where $\varepsilon \equiv m/M$ is the ratio of electron mass to the ion mass and ν is the normalized collisional frequency. The system of equations should be completed by the Faraday's law

$$\frac{\partial b}{\partial t} = \frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x}. \qquad (7)$$

At this point, let us consider the case of a strong rarefaction wave, $n \ll n_0$ [$u \ll 1$ in (5)]. This will shorten the form of derivations without limiting the generality of the final conclusions. Using $\partial/\partial t = -u\partial/\partial x$ and Eq. (5) at $u \ll 1$, we integrate (7) across the wave front and obtain, for the sum of forces acting on electron fluid in Eq. (6),

$$-E_z + v_x b = F_0(b), \qquad F_0(b) \equiv -F(b)/2n_0u, \qquad (8)$$

$$F(b) = (b_0^2 - b^2)b + \beta(b_0^2 - b^2)^2 - \gamma(b_0^2 - b^2),$$
$$\beta \equiv -\frac{1}{2n_0^2u^2}\frac{dn_0u}{dz}, \qquad \gamma \equiv -\frac{1}{2n_0u}\frac{db_0^2}{dz}. \qquad (9)$$

Here both β and γ are positive. During the integration we have taken into account the boundary condition $F_0(b_0) = 0$ in the unperturbed plasma at the anode. Further substitution of Eqs. (3) and (8) into (6) results in the following form of Ohm's law:

$$\varepsilon\frac{n_0u}{n}\frac{\partial v_z}{\partial x} - \frac{\varepsilon}{n}\{b, v_z\} - \frac{\nu}{n}\frac{\partial b}{\partial x} + F_0(b) = 0, \qquad \{b, v_z\} \equiv \frac{\partial b}{\partial x}\frac{\partial v_z}{\partial z} - \frac{\partial b}{\partial z}\frac{\partial v_z}{\partial x}. \qquad (10)$$

We look for the solution of Eq. (10) in the form $b(x, z, t) = b(z, \xi)$, where $\xi \equiv [x - u(z)t]f(z)$. With this new variable and still for $u \ll 1$ the Eqs. (3), (5), and (10) yield

$$\frac{\varepsilon f^2 n_0 u}{n}\left[\frac{d}{d\xi}\left(\frac{1}{n}\frac{db}{d\xi}\right) + \frac{g(b)}{n}\left(\frac{db}{d\xi}\right)^2\right] - \frac{\nu f}{n}\frac{db}{d\xi} + F_0(b) = 0, \qquad (11)$$

$$g(b) = \frac{2\gamma}{b_0^2 - b^2 + 2n_0u^2} - \frac{4\beta(b_0^2 - b^2)}{b_0^2 - b^2 + 2n_0u^2} + \chi(z), \qquad (12)$$

$$\chi(z) \equiv -\frac{1}{n_0(z)u}\frac{d\ln f(z)}{dz}. \qquad (13)$$

We further introduce a new variable, $d\zeta \equiv d\xi \times n/(n_0uf\sqrt{\varepsilon})$, and a new function, $Y(b) \equiv db/d\zeta$. Our main Eq. (11) then appears as follows:

$$\frac{dY^2}{db} + 2g(b)Y^2 - 2\eta Y = F(b), \qquad (14)$$

where $\eta = \nu/\sqrt{\varepsilon}$. We further consider the case of small collisions, $\eta \to 0$.

Ultimately, we came from Eq. (6) to Eq. (14) and should define now the boundary conditions. First, the sum of forces $F(b)$ must be equal to zero at both superconducting electrodes in order to prevent the current j_z from rising without limitations in (6). Besides Eq. (5) signifies $Y(b)|_{C,A} = db/d\zeta = 0$ and from Eq. (14) we also have that $dY^2/db|_{C,A} = 2d^2b/d\zeta^2 = 0$. Indeed, Eq. (14) may have a solution satisfying all these conditions:

$$Y^2(b) = e^{-G(b)}\int_{b_{\min}}^{b} F(b')e^{G(b')}db' \qquad (15)$$

with $G(b) \equiv \int 2g(b)db$. In the region sufficiently far from the anode, $b_0^2 - b^2 \gg n_0 u^2$ we have using (12)

$$G(b) = -2(4\beta - \chi)b + \frac{2\gamma}{b_0}\ln\left(\frac{b_0 + b}{b_0 - b}\right). \quad (16)$$

Thus, $G(b) \approx \alpha b + C$ and the integral (15) can be easily taken representing the analytical solution of our problem. $Y(b)|_C = 0$ automatically in (15), and the proper choice of χ function [note that $f(z)$ can be arbitrary in (13)] will further allow us to satisfy $Y^2(b_0) = dY^2/db = 0$ at the anode. As the final expression acquires quite a bulk form we do not present it here.

Interestingly, the behavior of the function $Y(b)$ can be analyzed without direct calculation of the integral (15). Starting with zero value at the cathode, the function $Y(b)$ then rises [$db/d\zeta|_C > 0$, see (3) with $v_z > 0$] and should have a maximum in the midgap. From Eq. (15) this signifies that $F(b)$, having zero value at the cathode, must change its sign from positive to negative and it becomes again zero at the anode. Indeed, we can find three real positive roots of Eq. (9)

$$b = b_0, \qquad b = b_{f,\min} = \frac{1}{2\beta} \pm \left(\frac{1}{4\beta^2} - \frac{\gamma}{\beta} + b_0^2\right)^{1/2}. \quad (17)$$

The condition $b_0 > b_f > b_{\min} > 0$ ($b_{\min}$ at the cathode, b_f at the front and b_0 at the anode) is equivalent (as $\beta > 0$) to

$$\beta b_0^2 < \gamma < \beta b_0^2 + \frac{1}{4\beta}, \qquad \gamma > b_0, \qquad \beta > \frac{1}{2b_0}. \quad (18)$$

The inequalities (18) allow us already to construct the rarefaction profile for some initial distributions $n_0(z)$ and $b_0(z)$. To demonstrate this, let us suppose that $u(z) \approx$ const and, in the first order,

$$n_0(z) \approx 1 - z/h_1, \qquad b_0(z) \approx 1 - z/h_2. \quad (19)$$

For each plasma length h one can find a number of values u, h_1, and h_2 satisfying the inequalities (18) at all the axial positions. Let us choose the maximum velocity, $u_{\max}$, which can be realized for given h. In this case, at the limit of validity of the inequalities (18), the following estimates can be kept:

$$h_1^{\min} \approx 2h, \qquad h_2^{\min} \approx 3h, \qquad u_{\max} \approx \frac{1}{3h}. \quad (20)$$

It is easy now to plot the profile of Eq. (15). We take, for instance, $h = 10$, $h_1 = 21$, $h_2 = 31.7$, and $u = 1/32$ in (20) and thus satisfy Eq. (18). For each given z, $0 < z < h$, there exists the value of χ, which puts the integral of Eq. (15) to zero at the anode. The resulting dependence $\chi(z)$ is presented in Fig. 2(a). The finally appearing wave profile is illustrated by Fig. 2(b) for some axial position.

Thus, Eq. (14) does correspond to a regular rarefaction wave with $b < b_0$ and $u > 0$ [$n < n_0$ in (5)]. For the magnetic field front width we find $\Delta\zeta \sim 1$, $\Delta x \sim \sqrt{\varepsilon/n}$ that corresponds to the size of $\delta \sim c/\omega_{pe}$ in dimensional units, i.e., is defined by the electron inertia, as it should be. It can be shown that the wave still exists in the collisional case $\eta \neq 0$ in Eq. (14). Similar to the compression waves [8], the front width is defined by collisions and widens with the increase of parameter η. The density drop is independent of plasma collisionality, as it follows from Eq. (5), and tends here to $n_0/n \to 430$.

Concluding with the collisionless case, the procedure for obtaining Figs. 2(a) and 2(b) can be repeated for other z. Together with definitions of the variables ζ and ξ, this provides us with the picture of Figs. 3(a) and 3(b). The wave arises due to acceleration of the ions, V_x, by the Hall electric field in the current layer near the cathode ($t = t_1$). It propagates further through the quasineutral magnetized plasma and can create a higher than 1 order of magnitude density drop ($t = t_2$).

The magnetic field is brought into the plasma being frozen in the electron component, which has the velocity v_e, Figs. 3(a) and 3(b). As the density decreases along electron trajectories, the specific magnetic flux, $v_z b$, exiting the plasma region is greater than that entering it. This corresponds to shrinking of the contour ⊗ with b_0,

$$-\frac{d\varphi}{dz} = \frac{d}{dz}\int_0^\Delta v_z b dx = \frac{d\dot{\Psi}}{dz} = -u\int_0^\Delta \frac{\partial b}{\partial x}dx \approx -ub_0 < 0, \quad (21)$$

and, therefore, supports a rarefaction wave having velocity u. Here φ is the voltage drop and $\dot{\Psi}$ is the magnetic flux change rate. Subsequently, the magnetic field has to be frozen out of the electrons and exits the system. In practice [2,5–7], when the described Hall plasma could be used for current switching into a load connected downstream, this event must happen and results in appearance of the load current, $J_d = -b_c(h) = \varphi(h)/R$, where R is the load resistance, and $b_c(z) \to b_{\min}(z)$ is provided by Eq. (17). In our consideration of the problem,

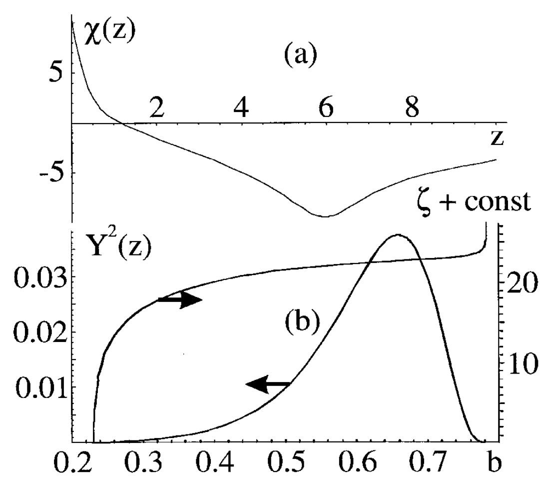

FIG. 2. (a) Calculated function $\chi(z)$ (see the text). (b) Corresponding solution (15) at $z = 7$ satisfying the necessary boundary conditions. Here $b_{\min} \approx 0.223$, $b_f \approx 0.354$, and $b_0 \approx 0.779$.

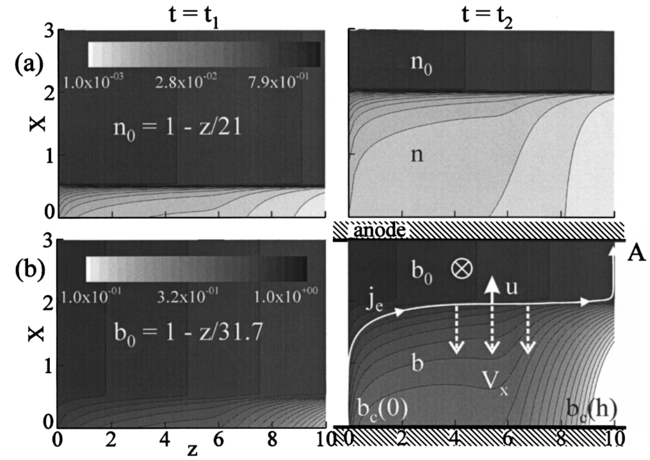

FIG. 3. Calculated two-dimensional density (a) and magnetic field (b) maps for the initial profiles of Eqs. (19) and (20) and for two time moments. Here x and z are dimensionless coordinates, v_e corresponds to the plasma electron current, and $b_c(z)$ is the magnetic field of the cathode part of the total current through the plasma region.

the electrodes are far enough from the wave front, but the flux freezing out is implicitly assumed and may occur, for example, in collisional anode plasma [4] [point A in Fig. 3(b)].

The magnetic energy is initially supplied to the plasma by the generator and can be calculated using the obtained solution $b(x, z)$. A part of this energy goes to the kinetic energy of ions accelerated in the Hall electric field, $V_x^2/2 = |\varphi|$. Thus, Eq. (5) with $b = b_c$ yields the voltage value, which tends to its maximum at $b_c = b_{\min}$ as the wave advances, see Figs. 3(a) and 3(b). The ion kinetic energy is further dissipated at the cathode and the associated positive charge reduces the cathode current, $J_c = -b_c$ [$b_c(z)$ is a decreasing function]. The rest of the energy in the system includes the Poynting flux, $\varphi(h)b_c(h)$, towards the load and the earlier mentioned dissipation in the triple point A. Consideration of the total energy balance with all the listed parts provides with a quantitative value for this dissipated energy. We postpone more details of this discussion to a later publication.

Equation (21) allows also a simple estimate for the front velocity, $u \sim \langle b^2/2n \rangle / b_0 h \sim 1/2h$, close to the result of Eq. (20). In dimensional units, this velocity appears as the fundamental Hall velocity of Eq. (1), corresponding here to expulsion of the magnetic field. Existence of such expulsion was discovered before [11] for collisional plasmas with motionless ions, similarly to how Eq. (1) was obtained. We confirm this possibility also for a zero-resistivity plasma by keeping the electron inertia term, and a slower magnetic field evolution than that appearing in [11] leads to considerable plasma motion in this case.

The initial assumptions, $c/\omega_{\rm pe} \ll c/\omega_{\rm pi} \ll h$ (see Fig. 1 and the nearby text), are met in quite broad range of laboratory or space plasmas [4]. Our main conclusion, while contradicting the universal conception from the conventional hydro [10], could explain the experimentally observed unusual pulsed-plasma behavior [5–7]. In these experiments, the magnetic field flowed into the plasma at considerably higher axial velocity than that of the ions, supporting applicability of Eq. (1) and of the system (2)–(7). Acquisition by the ions of much higher radial velocities than the axial ones $|V_x| \gg |V_z|$, that leads through the solution of the derived Eq. (14) and from Eq. (5) to rarefaction in our model, is not in contradiction with experimental observations either. Finally, the analysis (15)–(18) proposes the traveling profiles of Figs. 3(a) and 3(b) for understanding the observed subsequent plasma density drop.

In conclusion, this Letter states that the Hall plasmas can support rarefaction shocks. Unlike the conventional gas and magnetohydrodynamics, where such jumps with $n < n_0$ are not stable and decay [10], the discovered waves can be regular and their stationary space profile has been derived. For the collisionless case and large discontinuities, $n \ll n_0$, we investigated an example of the fastest wave existing for some realistic initial profiles of plasma density and magnetic field. Further theoretical work will address the issue of the rarefaction wave influence on the macroscopic Ohm's low (voltage-to-current dependence) for these plasmas.

[1] See *Proceedings of the IEEE International Pulsed Power Conferences* (http://www.ieee.org) and *Proceedings of the 12th IEEE International Conferences on High-Power Particle Beams*, edited by M. Markovits and J. Shiloh (IEEE Catalog No. 98EX103, Haifa, 1998).
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