Effects of Forcing in Three-Dimensional Turbulent Flows

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We present the results of a numerical investigation of three-dimensional homogeneous and isotropic turbulence, stirred by a random forcing with a power-law spectrum, $E_f(k) \sim k^{3-y}$. Numerical simulations are performed at different resolutions up to 512^3. We show that at varying the spectrum slope y, small-scale turbulent fluctuations change from a *forcing independent* to a *forcing dominated* statistics. We argue that the critical value separating the two behaviors, in three dimensions, is $y_c = 4$. When the statistics is forcing dominated, for $y < y_c$, we find dimensional scaling, i.e., intermittency is vanishingly small. On the other hand, for $y > y_c$, we find the same anomalous scaling measured in flows forced only at large scales. We connect these results with the issue of *universality* in turbulent flows.

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The effects of both external forcing mechanisms and boundary conditions on small-scale turbulent fluctuations have been the subject of many theoretical, numerical, and experimental studies [1,2]. The 1941 theory of Kolmogorov [1] is based on the assumption of *local* isotropy and homogeneity, that is, any turbulent flow, independently on the injection mechanism, recovers universal statistical properties, for scales small enough (and far from the boundaries). Indeed, experiments and numerical simulations give strong indications that Eulerian isotropic/anisotropic small-scale velocity statistics are pretty independent of the *large-scale* forcing mechanisms [3–6]. Still, we lack a firm understanding for these evidences. Here many of the powerful tools developed to discuss similar problems in systems at statistical equilibrium are not applicable. For that reason, understanding universality in Navier-Stokes equations may have important feedback on a broader class of out-of-equilibrium systems. From the theoretical point of view, precious hints arise from *linear* problems, such as passive scalar or passively advected magnetic fields. For the class of Kraichnan models, anomalous scaling has been shown to be associated to statistically stationary solutions of the unforced equations for correlation functions [7]. Scaling exponents are consequently universal with respect to the injection mechanisms. Concerning nonlinear problems, such as the Navier-Stokes case, analytical results have been often pursued by means of the renormalization group (RG) [8,9]. In the RG framework, turbulence is stirred at all scales by a self-similar Gaussian field, with zero mean and white noise in time. The two-point correlation function in Fourier space is given by

$$\langle f_i(\boldsymbol{k}, t) f_j(\boldsymbol{k}', t')\rangle \propto k^{4-d-y} P_{ij}(\boldsymbol{k}) \delta(\boldsymbol{k} + \boldsymbol{k}') \delta(t - t'). \quad (1)$$

Here $P_{ij}(\boldsymbol{k})$ is the projector assuring incompressibility and d is the spatial dimension (always assumed to be $d = 3$ hereafter). The influence of the stirring mechanism at small scales is governed by the value of the slope y. We go from a situation when the forcing has a strong input at all scales, $y \sim 0$ originally investigated in [8], to a quasi-large-scale forcing when $y \to \infty$. Renormalization group calculations, based on a y expansion, predict a power-law energy spectrum $E(k) \sim k^{1-2y/3}$ in the domain $\eta \ll k^{-1} \ll L$, where η is the viscous scale of the system, and for $y \ll 1$. Notice that the Kolmogorov value $E(k) \sim k^{-5/3}$, describing experimental turbulent flows stirred by a large-scale forcing, is obtained for $y = 4$, i.e., quite far from the perturbative region where the RG calculations are under control. The Kolmogorov spectrum can be obtained, however, by means of a simple dimensional analysis, still within the same framework [10]. Extension of the RG formalism to finite y values, up to $y = 4$, have been attempted with a different kind of approximation [11,12] although in a range where convergence of the RG expansion is not granted anymore [13]. As for the numerical simulations, in [14] the problem has been investigated for various y values, and it has been shown that, for $y = 4$, results are in good agreement with the picture of large-scale forced turbulence, while for $y < 4$ the situation becomes less clear. However, the low numerical resolution used in [14] makes these results far from being conclusive. A similar study has also been performed in the case of shell models for turbulence [15].

Beside the issue connected to the RG approach, there exists a whole set of interesting questions concerning turbulent flows with a power-law forcing. To what extent are small-scale fluctuations sensitive to the injection mechanism? Does there exist a critical value y_c separating different regimes? Can we observe anomalous scaling in the forcing dominated case $y \sim 0$? Let us suppose, for example, that there exists a finite y_c, beyond which small-scale statistics is forcing independent: this would rule out any attempt to control intermittency analytically, by means of a perturbative approach which starts from forcing dominated turbulent solutions at $y \ll y_c$.

Hints on the problem can also come from the study of the one-dimensional Burgers equation in presence of a power-law forcing. In [16], a numerical study was presented showing that there is a critical value of the forcing slope, such that the velocity field passes from the usual bifractal statistics (observed in large-scale forced Burgers flows), to a statistics affected by the forcing. Surprisingly, also in the forcing dominated regime, a nontrivial (multifractal?) scaling was observed in [16]. A rigorous understanding of the mechanism leading to this result is, however, still missing: we will come back to this point later on, after having presented our results.

In the sequel, we address the problem of the small-scale statistical properties of three-dimensional turbulent flows at varying parameter y. We show that at crossing $y_c = 4$ there exist two regimes for the velocity field statistics with well defined and different scaling properties. At difference from what was shown in [14], we study the behavior of the whole velocity probability density function (PDF) at changing the scale for $y < y_c$, showing that in this regime velocity fluctuations are Gaussian, i.e., there is no leading anomalous contribution. This is far from being trivial because for $y \gg 0$ ($y = 3.5$ here) RG calculations are well beyond their range of validity. Only direct numerical simulations can make firm quantitative statements. Furthermore, in the range $y > y_c$ we find that the velocity PDF is intermittent, anomalous scaling being in good quantitative agreement with that found with large-scale forcing. This contradicts what was found in [14] where, probably because of finite Reynolds effects due to low resolution, different scaling properties were claimed with respect to the usual large-scale Navier-Stokes system. Indeed, subleading contributions are unavoidable when a power-law forcing is present: only high resolution can help to disentangle them.

We solved the Navier-Stokes equations with a second-order hyperviscous dissipative term $\propto \nu\Delta^2$, which yields a larger inertial range without affecting scaling properties [17]. Temporal integration has been carried over for about 20–30 large-eddy turnover times. We performed various experiments, at resolutions 128^3, 256^3, and 512^3, corresponding to a maximum Taylor's Reynolds number equal to $\mathrm{Re}_\lambda = 220$ for the 512^3 run. As for the stirring force, we specialized in the two following cases, one for each regime: the first with $y = 3.5 < y_c$, the second with $y = 6 > y_c$. The range of the forcing, in Fourier space, extends down to the maximum resolved wave number. As we are always confined in a finite box, we neglect here possible subtle behaviors due to infrared divergences in the injection mechanism. We also show results obtained with an analytical large-scale forcing, i.e., a forcing with support on only a few wave numbers. This is the equivalent to $y \to \infty$, in previous notation.

We start considering what happens to the system when the slope of the forcing is changed. It is instructive to consider the equation for the energy flux through the wave number k: $\Pi(k) \equiv -\pi k^2 \int \Im[\langle(\boldsymbol{k}\cdot\hat{\boldsymbol{v}}(\boldsymbol{q}))(\hat{\boldsymbol{v}}(\boldsymbol{k})\cdot\hat{\boldsymbol{v}}(\boldsymbol{p}))\rangle + \langle(\boldsymbol{k}\cdot\hat{\boldsymbol{v}}(\boldsymbol{p}))(\hat{\boldsymbol{v}}(\boldsymbol{k})\cdot\hat{\boldsymbol{v}}(\boldsymbol{q}))\rangle]d\boldsymbol{p}d\boldsymbol{q}$, where the three wave vectors satisfy $\boldsymbol{k}+\boldsymbol{p}+\boldsymbol{q}=0$, and the symbol $\Im$ stands for the imaginary part. Such an equation is equivalent to the Kármán-Howarth equation in physical space; it states that in a stationary, isotropic, and homogeneous flow, the contribution to the energy flux $\Pi(k)$ due to the nonlinear terms balances the total energy input from the injection mechanism (see [1]):

$$\Pi(k) \sim \int_{k_0<|\boldsymbol{k}|<k} \Re(\langle \boldsymbol{f}(\boldsymbol{k})\boldsymbol{v}(-\boldsymbol{k})\rangle)d\boldsymbol{k}, \tag{2}$$

where $k_0 \sim 1/L$, the symbol $\Re$ stands for the real part, and where we have neglected dissipative effects. For the special class of forcings (1), the right-hand side of (2) can be further simplified to $\Pi(k) \sim \int_{k_0<|\boldsymbol{k}|<k}\langle|\boldsymbol{f}(\boldsymbol{k})|^2\rangle d\boldsymbol{k}$. From (1) the forcing spectrum is $E_f(k) = \langle|\boldsymbol{f}(\boldsymbol{k},t)|^2\rangle \sim k^{3-y}$. It follows that for $y \geq 4$, the energy flux is constant in Fourier space for $kL \gg 1$ (up to logarithmic corrections for $y = 4$). In other words, the energy injection is dominated by the small wave-number region in the integral (2). In this case we expect to be very close to the typical experimental situation of turbulence with a large-scale, analytical forcing: energy is transferred downscale via an intermittent cascade. Coherently, the third order longitudinal structure function, $S^{(3)}(r) \equiv \langle[(\boldsymbol{v}(\boldsymbol{x}+\boldsymbol{r}) - \boldsymbol{v}(\boldsymbol{x}))\cdot\hat{\boldsymbol{r}}]^3\rangle$, follows a linear behavior in r as predicted by the 4/5 law [1]. For $y < 4$, the energy flux no longer saturates to a constant value as a function of k. The integral in (2) now becomes ultraviolet dominated. The direct input of energy from the forcing mechanism affects inertial range statistics in a self-similar way, down to the smallest scales where dissipative terms start to be important. In this situation, we get for the energy flux $\Pi(k) \sim k^{4-y}$, with a constant prefactor which depends on the ultraviolet cutoff. The corresponding scaling behavior for the third order structure function is now given by $S^{(3)}(r) \sim r^{y-3}$.

What about higher order statistics? One is tempted to guess that for $y \geq 4$ the fluctuations induced by the injection mechanism are always subleading, anomalous scaling being the result of the cascade mechanism driven by the nonlinear terms of the equations of motion. If so, for $y > 4$ we should fall in the same class of "universality" of turbulence generated with large-scale forcing, i.e., small-scale velocity fluctuations should be universal. Therefore, as far as $y > 4$, longitudinal structure functions should scale as:

$$S^n(r) \equiv \langle[(\boldsymbol{v}(\boldsymbol{x}+\boldsymbol{r}) - \boldsymbol{v}(\boldsymbol{x}))\cdot\hat{\boldsymbol{r}}]^n\rangle \sim r^{\zeta_\infty^{(n)}}. \tag{3}$$

In (3), we have denoted with ζ_∞^n the scaling exponents measured with an analytical, large-scale forcing.

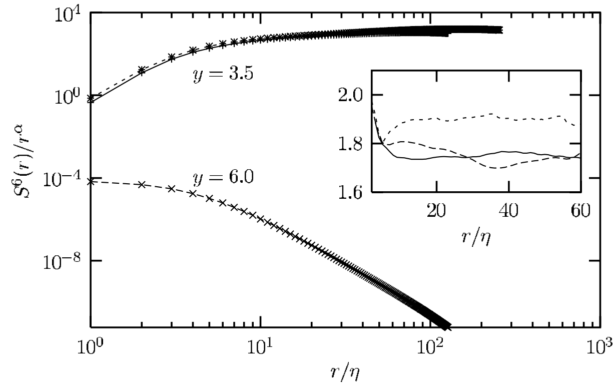

FIG. 1. Log-log plot of the compensated sixth-order structure function $S^{(6)}(r)/r^{\alpha}$. The two top curves are for $y = 3.5$ at the two resolutions 256^3 and 512^3: they are compensated with the dimensional scaling (4), i.e., with an exponent $\alpha = \zeta^{(6)}_{y=3.5} = 1$. The bottom curve refers to the case $y = 6$, at the resolutions 256^3, and is also compensated with the exponent for the scaling (4), $\alpha = \zeta^{(6)}_{y=6} = 6$. Clearly the matching with the dimensional exponent is not the correct one in the case $y = 6$. Inset: local slopes of the extended self-similarity (ESS) curve [18], for $S^{(6)}(r)$ vs $S^{(3)}(r)$, at varying r. Top curve refers to the case $y = 3.5$, and the two bottom curves refer to the cases $y = 6$ and $y \to \infty$. The dimensional scaling would correspond to the value 2.

On the other hand, for $y < 4$, energy is directly injected in the inertial range. Here a dimensional matching with the forcing gives a scaling behavior which is always leading with respect to that predicted in the $y > 4$ range (3). We expect now that anomalous scaling disappears, everything being dominated by the Gaussian energy input at all scales. By the simple dimensional argument connecting the scaling of structure functions to that of the external forcing, for the range $y < 4$ we have

$$S^{(n)}(r) \sim r^{\zeta_y^{(n)}} \quad \text{with } \zeta_y^{(n)} = \frac{n}{3}(y - 3). \tag{4}$$

In Fig. 1 we show the sixth order structure function, $S^{(6)}(r)$ for the two cases $y = 3.5$ and $y = 6$, compensated with the dimensional prediction (4) given by the matching with the forcing.

As it is clear, only for $y = 3.5$ do the statistics follow the forcing injection obtaining a nice compensation. On the other hand, for $y = 6$ the statistics is much closer to that usually measured with an analytical large-scale forcing. This is quantitatively confirmed by the inset in Fig. 1, where we plot the logarithmic derivatives of $S^{(6)}(r)$ vs $S^{(3)}(r)$, for the two cases $y = 3.5$ and $y = 6$, together with the results of the simulation with a large-scale, smooth forcing corresponding to $y \to \infty$. Here the local slopes for $y = 6$ and $y \to \infty$ fluctuate around the same value, compatible with those reported in literature [2,4], while the local slope for $y = 3.5$ is different and tends to the expected dimensional value. Values of all scaling exponents obtained in the simulations are summarized in Table I. Let us notice that the measured exponents for the $y = 3.5$

TABLE I. Scaling exponents in ESS, of the curves $S^{(n)}(r)$ vs $S^{(3)}(r)$, extracted from the following numerical simulations: $y = 3.5$, at resolution 256^3 and 512^3; $y = 6$, at resolution 256^3; $y \to \infty$, at resolution 512^3. The first row describes the dimensional values: $\zeta_y^{(n)}/\zeta_y^{(3)} = n/3$.

$\zeta_y^{(n)} \backslash n$	1	2	4	5	6
y_d	0.333	0.666	1.33	1.66	2.00
$y = 3.5$	0.34(1)	0.67(1)	1.31(2)	1.62(2)	1.93(3)
$y = 6$	0.36(1)	0.69(1)	1.28(2)	1.53(3)	1.75(4)
$y \to \infty$	0.36(1)	0.69(1)	1.27(2)	1.52(3)	1.75(4)

case are very close to the nonintermittent, dimensional prediction. Only for high order moments (i.e., $n = 6$) is there a *small* deviation from the expected value $\zeta^{(6)}/\zeta^{(3)} = 2$. To quantify the level of intermittency at changing the scale, we also plot the PDF of velocity increments at different scales, normalized to have unit variance. Figure 2 shows the PDFs, in the case $y = 6$, for three different separations: $r_1 = 34\eta$ and $r_2 = 74\eta$ in the inertial range and $r_3 = 114\eta$ in the energy containing range. The three curves have larger tails than a Gaussian distribution and have an intermittent behavior; i.e., they cannot be superposed. In Fig. 3 we show the PDFs, for the case $y = 3.5$, at the same separations (r_1, r_2, r_3). Now the three curves are almost indistinguishable and show a very good rescaling, a signature of the absence of intermittent effects. Only for negative increments is a very tiny discrepancy measured. It is hard to say whether this is a robust effect or a spurious Reynolds dependent phenomenon. We will come back to this issue later in the conclusions.

A dramatic difference at crossing the $y = y_c$ value is also observed in the energy dissipation statistics. For both cases $y = 3.5$ and $y = 6$, we measured the PDF of the coarse-grained energy dissipation $\varepsilon_r(\boldsymbol{x}) = V_r^{-1} \int_{V_r(\boldsymbol{x})} \varepsilon(\boldsymbol{x} + \boldsymbol{r}) d\boldsymbol{r}$, where ε is the rate of dissipation for unit volume, and the volume $V_r(\boldsymbol{x})$ is centered at $\boldsymbol{x}$ and has characteristic length scale $r \ll L$. In Fig. 4 we compare the PDFs $\mathcal{P}(\varepsilon_r)$ at the scale $r = 8\eta$. Here the results

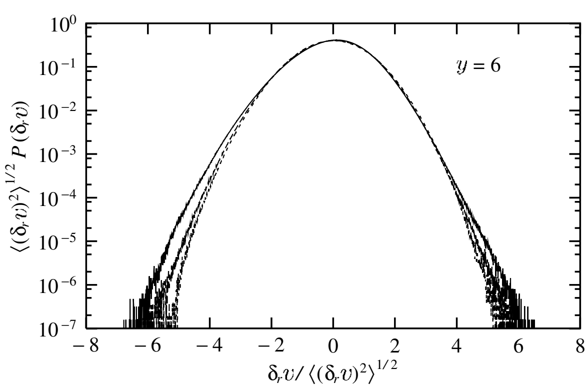

FIG. 2. PDF of the velocity increments, for $y = 6.0$, for three separations: $r_1 = 34\eta$ and $r_2 = 74\eta$ in the inertial range and $r_3 = 114\eta$ in the energy containing range.

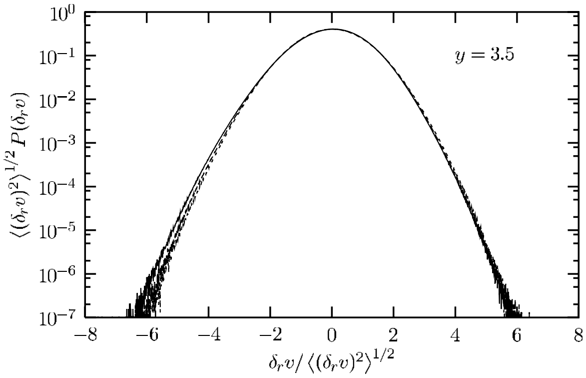

FIG. 3. PDF of the velocity increments, for $y = 3.5$. Scales are the same as the previous figure: $r_1 = 34\eta$, $r_2 = 74\eta$, and $r_3 = 114\eta$.

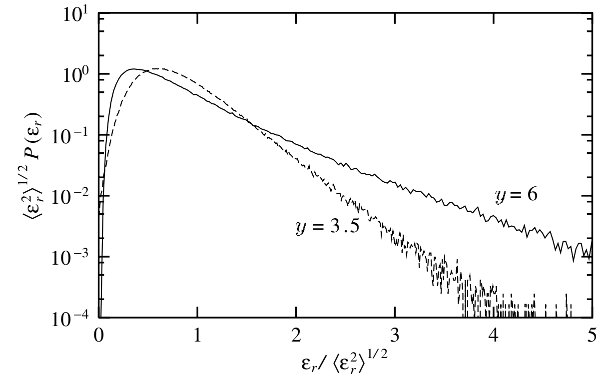

FIG. 4. PDF of the coarse-grained energy dissipation $\mathcal{P}(\varepsilon_r)$, for both $y = 3.5$ and $y = 6.0$, at the scale $r = 8\eta$.

are even more impressive as the shape change is particularly strong.

In this Letter, we have presented evidence that turbulent small-scale fluctuations, in the presence of a direct injection of energy at all scales, undergo a transition for $y_c = 4$. The first regime, for $y > y_c$, is mainly forcing independent: small-scale fluctuations develop anomalous scaling in agreement with that observed in experiments and/or numerics obtained with a large-scale forcing. This is a stringent test of turbulence universality: even if directly affected by the injection of energy, small-scale fluctuations show a robust behavior.

Things change abruptly at the critical value of y_c, where the direct injection of energy becomes the dominant effect in the inertial range. In this second regime, corresponding to $y < y_c$, small-scale fluctuations get closer and closer to a Gaussian statistics and intermittency disappears.

Before concluding, we discuss two possible mechanisms which could partially disprove the last statement. First, even when $y < y_c$, we may imagine that the intermittent energy cascade dominating the statistics for $y > y_c$ might show up. For example, we may have that for high order moments the forcing dominated solutions become subleading with respect to those associated to the cascading mechanism. In such a case, the loss of rescaling in the PDF's tail in Fig. 2 may be due to the survival of these rare anomalous fluctuations. Second, even more complex is the scenario proposed in [16], where the possibility to have a forcing dependent multiscaling statistics, when $y < y_c$, is conceived. This is not the case for the linear Kraichnan models [7], where forcing dependent solutions are always dimensionally scaling. The main difference is that, in the Navier-Stokes problem, the hierarchy of equations for correlation functions is unclosed: one cannot solve it for a single order independently of all the others. In Navier-Stokes, being the low order moments always forcing dominated for $y < y_c$, one may observe some forcing dependency also on high order fluctuations via their coupling with low order correlation functions.

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[1] U. Frisch, *Turbulence: The Legacy of A. N. Kolmogorov* (Cambridge University Press, Cambridge, 1995).
[2] K. R. Sreenivasan and R. A. Antonia, Annu. Rev. Fluid Mech. **29**, 435 (1997).
[3] X. Shen and Z. Warhaft, Phys. Fluids **14**, 370 (2002).
[4] T. Gotoh, D. Fukayama, and T. Nakano, Phys. Fluids **14**, 1065 (2002).
[5] L. Biferale, G. Boffetta, A. Celani, A. Lanotte, F. Toschi, and M. Vergassola, Phys. Fluids **15**, 2105 (2003).
[6] L. Biferale, E. Calzavarini, F. Toschi, and R. Tripiccione, Europhys. Lett. **64**, 461 (2003).
[7] G. Falkovich, K. Gawędzki, and M. Vergassola, Rev. Mod. Phys. **73**, 913 (2001).
[8] D. Forster, D. R. Nelson, and M. J. Stephen, Phys. Rev. A **16**, 732 (1977).
[9] U. Frisch and J. D. Fournier, Phys. Rev. A **17**, 747 (1978).
[10] C. De Dominicis and P. C. Martin, Phys. Rev. A **19**, 419 (1979).
[11] V. Yakhot and S. A. Orszag, Phys. Rev. Lett. **57**, 1722 (1986).
[12] L. Ts. Adzhemyan, N. V. Antonov, M. V. Kompaniets, and A. N. Vasil'ev, Int. J. Mod. Phys. B **17**, 2137 (2003).
[13] G. Eyink, Phys. Fluids **6**, 3063 (1994).
[14] A. Sain, Manu, and R. Pandit, Phys. Rev. Lett. **81**, 4377 (1998).
[15] B. Mazzi, F. Okkels, and J. C. Vassilicos, Eur. Phys. J. B **28**, 243 (2002).
[16] F. Hayot and C. Jayaprakash, Phys. Rev. E **56**, 4259 (1997); **56**, 227 (1997).
[17] V. Borue and S. A. Orszag, J. Fluid Mech. **306**, 293 (1996).
[18] R. Benzi, S. Ciliberto, C. Baudet, G. Ruiz Chavarria, and R. Tripiccione, Europhys. Lett. **24**, 275 (1993).