Comment on "Antispiral Waves in Reaction-Diffusion Systems"

In a recent Letter, Gong and Christini [1] provide evidence for inwardly rotating spiral waves (antispiral waves) in simulations of reaction-diffusion (RD) models. They present a theoretical argument within the complex Ginzburg-Landau equation (CGLE):

$$\partial_t W = W - (1 + i\alpha)|W|^2 W + (1 + i\beta)\nabla^2 W. \quad (1)$$

They find that antispiral waves (AS) occur for $\alpha\beta \geq 0$ in the parameter regions $\alpha > \beta \geq 0$ and $0 \geq \beta > \alpha$. It is suggested to derive the CGLE for a given RD model and apply the criteria for AS. This Comment clarifies two crucial points not treated in [1]: (i) The criterion for AS emergence in the CGLE can be derived analytically using earlier results. (ii) Regions for AS in the CGLE and in the corresponding RD model are different; the correct CGLE criterion for occurrence of AS in a RD model is $\alpha < \beta$ in contrast to the results in [1].

Consider a general RD model in two dimensions (2D):

$$\partial_{\tilde{t}} \tilde{\mathbf{u}} = \tilde{\mathbf{f}}(\tilde{\mathbf{u}}, \mu) + \mathcal{D}\nabla^2 \tilde{\mathbf{u}}, \quad (2)$$

where $\tilde{\mathbf{u}}(\tilde{\mathbf{x}}, \tilde{t})$ is a vector of concentrations and μ is a control parameter. Near a Hopf bifurcation at $\mu = \mu_C$ of a state $\tilde{\mathbf{u}}_0$ defined by $\tilde{\mathbf{f}}(\tilde{\mathbf{u}}_0) = 0$ and with critical frequency Ω and eigenvector $\tilde{\mathbf{u}}_1$, the vector $\tilde{\mathbf{u}}$ may be decomposed as $\tilde{\mathbf{u}}(\tilde{\mathbf{x}}, \tilde{t}) = \tilde{\mathbf{u}}_0 + \tilde{\mathbf{u}}_1 A(\mathbf{x}, t) e^{i\Omega\tilde{t}} + \tilde{\mathbf{u}}_1^* A^*(\mathbf{x}, t) e^{-i\Omega\tilde{t}}$. The CGLE Eq. (1) describes the evolution of slow modulations $A(\mathbf{x}, t) = \sqrt{\epsilon} W(\mathbf{x}, t) e^{ic_0 t}$ of a fast homogeneous oscillation [2]. Here, $\epsilon = (\mu - \mu_c)/\mu_c \ll 1$ and c_0 is an overall frequency shift. The CGLE coordinates are $\mathbf{x} = \sqrt{\epsilon}\tilde{\mathbf{x}}$ and $t = \epsilon\tilde{t}$. Frequencies ω in the CGLE result only in a small correction of order $\epsilon\omega$ to the original frequency Ω.

Spiral waves of the 2D CGLE have the form $W(r, \theta, t) = F(r) e^{i[\theta + f(r,t)]}$ in polar coordinates (r, θ). For $r \to \infty$, the radial dynamics follow $F(r) \to \sqrt{1 - k_S^2}$ and $f(r, t) \to k_S r - \omega_S t = k_S(r - v_{\mathrm{ph}} t)$ with a selected wave number k_S uniquely determined by α, β and a frequency $\omega_S = \alpha + (\beta - \alpha)k_S^2$. Hagan constructed a nonlinear eigenvalue problem for $k_S(\alpha, \beta)$ [3]. Its result agrees well with the k_S selected by spirals in simulations of the 2D CGLE [2]. The results of [2,3] indicate that $k_S = 0$ if $\alpha = \beta$ and $k_S > 0$ ($k_S < 0$) if $\alpha < \beta$ ($\alpha > \beta$). The group velocity $v_{\mathrm{gr}} = d\omega_S/dk_S = 2(\beta - \alpha)k_S > 0$. The phase velocity $v_{\mathrm{ph}} = \omega_S/k_S$ can change sign, if either k_S or ω_S changes sign. Phase waves may travel outward ($v_{\mathrm{ph}} > 0$) or inward ($v_{\mathrm{ph}} < 0$) in the CGLE. Consequently, $v_{\mathrm{ph}} > 0$ corresponds to spirals and $v_{\mathrm{ph}} < 0$ to AS in the CGLE. The two curves $\omega_S = 0$ and $k_S = 0$ separate regions where antispirals and spirals appear in the CGLE (see Fig. 1). v_{ph} diverge for $k_S \to 0$, while $v_{\mathrm{ph}} = 0$ for $\omega_S = 0$. Figure 1 recovers the CGLE results of [1] and extends them to regions with $\alpha\beta < 0$.

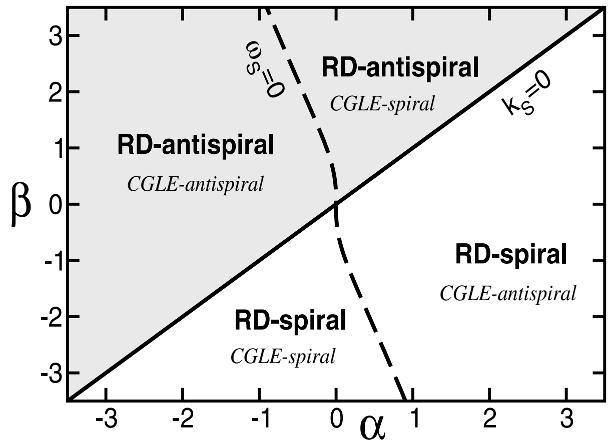

FIG. 1. Parameter space (α, β) of the CGLE is separated by the curves $\omega_S = 0$ and $k_S = 0$ in four subdomains. Bold (italic) text refers to the RD model (corresponding CGLE).

In the original RD model, the group velocity $\tilde{v}_{\mathrm{gr}} = \sqrt{\epsilon} v_{\mathrm{gr}} > 0$. For the phase velocity of waves, one obtains $\tilde{v}_{\mathrm{ph}} = [-\Omega + \epsilon(\omega_S - c_0)]/\tilde{k}_S \approx -\Omega/(\sqrt{\epsilon} k_S)$. Hence, $\tilde{v}_{\mathrm{ph}}$ changes sign only where k_S changes sign, i.e., for $\alpha = \beta$. Any ω_S in the CGLE is compensated by the fast frequency Ω; thus, an AS in the CGLE can represent a spiral in the corresponding RD model and vice versa (see Fig. 1). Altogether, AS occur in RD systems for positive $\tilde{k}_S$, i.e., for $\alpha < \beta$. The frequency $\Omega_{AS} = \Omega_0 - \tilde{k}_S^2(\beta - \alpha) < \Omega_0$, where Ω_0 is the bulk frequency, in agreement with experiments exhibiting AS in a chemical reaction [4].

Lutz Brusch, Ernesto M. Nicola, and Markus Bär
Max-Planck-Institut für Physik Komplexer Systeme
Nöthnitzer Strasse 38, 01187 Dresden, Germany

Received 9 April 2003; published 24 February 2004
DOI: 10.1103/PhysRevLett.92.089801
PACS numbers: 82.40.Ck, 05.45.–a, 47.54.+r, 89.75.Kd

[1] Y. Gong and D. J. Christini, Phys. Rev. Lett. **90**, 088302 (2003).
[2] I. S. Aranson and L. Kramer, Rev. Mod. Phys. **74**, 99 (2002).
[3] P. S. Hagan, SIAM J. Appl. Math. **42**, 762 (1982).
[4] V. K. Vanag and I. R. Epstein, Science **294**, 835 (2001).

