

Classical Scaling Theory of Quantum Resonances

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The quantum resonances occurring with δ -kicked particles are studied with the help of a fictitious classical limit, establishing a direct correspondence between the nearly resonant quantum motion and the classical resonances of a related system. A scaling law which characterizes the structure of the resonant peaks is derived and numerically demonstrated.

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Resonances are a widespread phenomenon both in quantum and in classical physics. However, quantum resonances are often purely quantal phenomena, not directly related to classical nonlinear resonances. Atom optics has made crucial aspects of the classical-quantum correspondence accessible to direct experimental observation. An outstanding example is the paradigmatic kicked rotor (KR) model, which served as a prototype of quantum and classical Hamiltonian chaos over almost three decades. This model has been experimentally realized by a technique first introduced by Raizen and co-workers [1], allowing for observation of its main physical properties. These are connected with the long-time evolution, and depend on the arithmetic nature of the effective Planck's constant (which is the Planck constant divided by a characteristic action) [2]. If the latter is sufficiently irrational, the rotor's energy saturates in time. This effect is known as dynamical localization. If the effective Planck's constant is a rational multiple of 4π ($=4\pi r/q$, with r, q integers) unbounded growth of energy typically occurs. These are known as quantum resonances and have no counterpart in the corresponding classical system. Besides confirming theoretically known facts, experiments have also introduced variants of the basic model, which have enriched the theory. A striking example was the experimental discovery of unexpected "quantum accelerator modes" due to effects of gravity [3].

Even in cases when no major variants are introduced, experiments inevitably involve deviations from the idealized theoretical model. The most obvious of these is that atoms move in lines and not in circles; though to some extent trivial, this fact imposes certain modifications on the standard KR theory, especially in the case of the quantum resonances [4]. Another experimental limitation is that experiments are confined to not too long times. Consequently, they can neither detect the high-order (high- q) quantum resonances nor address the basic, as yet unsolved, mathematical problem of what degree of irrationality is required for localization. Nevertheless, for

this reason they have encouraged fruitful theoretical analysis of the intermediate time regime, which had rarely been investigated in fine detail.

In this Letter we present experimentally relevant results concerning the manifestations of quantum resonances at finite observation times. Our analysis enlightens the connections between quantum and classical resonances. When the energy of the kicked atoms is measured after a fixed time, and the result is plotted versus the kicking period, resonances manifest in the form of peaks at the resonant values of the period (see Fig. 1). We describe the line shape of such resonant peaks and its dependence on the observation time and on the other relevant parameters. Our main result is a single-parameter scaling law for such peaks, which, in particular, implies an inverse square dependence of the peak width on time, and is probably related to the "sub-Fourier resonances"

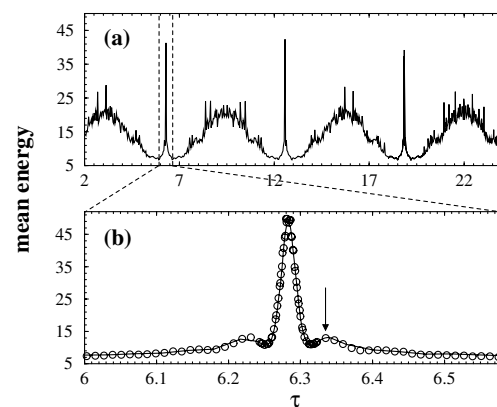


FIG. 1. (a) Mean energy after $t = 30$ kicks, vs the kicking period τ , for an ensemble of 10^5 δ -kicked atoms, with Gaussian initial momentum distribution ($\sigma \approx 2.7$) and $k = 0.8\pi$. (b) Quantum data are compared with the mean energies of an ensemble of 10^6 particles (circles), evolving under the ϵ -classical dynamics (5). The small peak on the right of the resonant spike is marked by an arrow for reference to Fig. 2. [The peaks appear lower in (a) because the used computational grid in τ had a lower resolution than in (b).]

recently detected in a different atom-optical context [5]. Our derivation is based on a quasiclassical analysis of the purely quantal resonances and their vicinity. This seemingly self-contradictory task is accomplished by establishing a direct correspondence between the quantum resonances and the classical resonances of a related system, which is *not* obtained in the usual classical limit of vanishing Planck's constant, but rather in the limit when the effective Planck's constant approaches its resonant value, which is of order unity. In this limit, the detuning of the driving period from resonance plays the role of the Planck constant. The same technique was used in [4] to explain the quantum accelerator modes observed in the presence of gravity. Here it is applied in the gravity-free case, leading to a different physical scenario. It can be adapted to the important case when decoherence is added [6].

The specific system we consider is described, in dimensionless units, by the Hamiltonian

$$\mathcal{H}(t') = \frac{\tau}{2} p^2 + k \cos(x) \sum_{t=-\infty}^{+\infty} \delta(t' - t), \quad (1)$$

where x is the coordinate, p is its conjugate momentum, t' is the continuous time variable, while t is an integer that counts the number of kicks. If x is an angle variable, confined to the interval $[0, 2\pi)$, then (1) is the Hamiltonian of the KR and p is the angular momentum; when quantized, it takes integer values only. The two dimensionless parameters that control the dynamics are the effective Planck's constant τ and the strength of the kick k . The first (second) is proportional (inversely proportional) to Planck's constant \hbar ; therefore, $k\tau = K$ is independent of \hbar . K is the stochasticity parameter of the standard map, which rules the classical dynamics defined by (1). The energy levels $\tau n^2/2$ (n integer) of the free rotor ($k = 0$), and differences between them, are integer multiples of $\tau/2$, while the frequencies of the driving potential are integer multiples of 2π ; consequently, quantum resonances of the kicked rotor (1) are found if $\tau/4\pi$ is rational [7]. These resonances correspond to the Talbot effect in optics and 4π corresponds to the Talbot time [8]. For most rational $\tau/4\pi$ they result in ballistic growth of p (quadratic growth of energy) with time. If x is not a coordinate on a circle but on the full line instead, then p is linear momentum and (1) yields the kicked particle (KP) model which is the one relevant for recent atom optics experiments. Since the driving potential is periodic in x , only transitions between values of p that differ by an integer are allowed. Consequently, the quasimomentum (QM) β (the fractional part of p) is conserved [4,6]. Standard Bloch theory allows for reduction of the KP dynamics to the dynamics of a bundle of rotors. At any given (and constant in time) value of β , the dynamics induced by (1) is formally that of a rotor and the one-period Floquet operator for this rotor is [4]

$$\hat{\mathcal{U}}_{\beta} = e^{-ik \cos(\hat{\theta})} e^{-i\tau(\hat{\mathcal{N}} + \beta)^2/2}, \quad (2)$$

where $\theta = x \bmod(2\pi)$, and $\hat{\mathcal{N}} = -id/d\theta$ is the angular momentum operator. If $\tau/4\pi$ is rational, the KP dynamics may exhibit asymptotic quadratic growth of the energy only for finitely many special values of the QM β . The particle is then in a spatially extended state [6]. Wave-packet dynamics, or incoherent ensemble averaging over momentum as is the case in experimental realizations [9], enforces *linear* growth of the energy as a function of time for rational $\tau/4\pi$ [4,6]. Away from resonances, the growth of energy in time is bounded owing to dynamical localization [2], so a scan of the particle energy vs τ after a fixed evolution time exhibits peaks at the resonant values of τ , as shown in Fig. 1. Such peaks rise and shrink as the observation time is increased. Their height is proportional to time, and their width to inverse *square* time, as our present analysis shows.

We restrict to the main resonances and to their vicinity, that is, to $\tau = 2\pi\ell + \epsilon$ where ℓ is a positive integer and ϵ is small compared to 2π . We denote $\tilde{k} = k|\epsilon|$, and $\hat{I} = |\epsilon|\hat{\mathcal{N}}$. Then, using the identity $e^{i\pi\ell n^2} = e^{i\pi\ell n}$, Eq. (2) may be rewritten as [4,6]

$$\hat{\mathcal{U}}_{\beta}(t) = e^{-i\tilde{k} \cos(\hat{\theta})/|\epsilon|} e^{-i\hat{\mathcal{H}}_{\beta}/|\epsilon|}, \quad (3)$$

where

$$\hat{\mathcal{H}}_{\beta}(\hat{I}, t) = \frac{1}{2} \text{sgn}(\epsilon) \hat{I}^2 + \hat{I}(\pi\ell + \tau\beta). \quad (4)$$

If $|\epsilon|$ is regarded as Planck's constant [10], then (3) is the formal quantization of either of the following classical maps:

$$\begin{aligned} I_{t+1} &= I_t + \tilde{k} \sin(\theta_{t+1}), \\ \theta_{t+1} &= \theta_t \pm I_t + \pi\ell + \tau\beta \bmod(2\pi), \end{aligned} \quad (5)$$

where \pm is the sign of ϵ . The small- $|\epsilon|$ asymptotic of the quantum β rotor is thus equivalent to a quasiclassical approximation based on the ϵ -classical dynamics (5). Here the term "classical" is not related to the $\hbar \rightarrow 0$ limit but to the limit $\epsilon \rightarrow 0$ instead, so the term " ϵ -classical" is used in the following. Note that the effective Planck's constant τ is *not small*, and important quantum symmetries are taken into account in the ϵ -classical description. The efficiency of this approximation is demonstrated in Fig. 1(b), where results of exact quantum simulations are compared to results obtained by (5), for an initial incoherent ensemble of particles, each in a momentum eigenstate (a plane wave) and with a Gaussian distribution of the momentum $p = n_0 + \beta$. For each particle in the ensemble, the map (5) with β equal to the QM of the particle was used to compute a set of trajectories started at $I = n_0|\epsilon|$ with homogeneously distributed $\theta \in [0, 2\pi)$. The final energies $\epsilon^{-2}I_t^2/2$ at $t = 30$ of the individual trajectories were averaged over θ, β, n_0 with the weights imposed by the initial ensemble. The average energy $\langle E_t \rangle = \epsilon^{-2}\langle I_t^2 \rangle/2$ is plotted vs $\tau = 2\pi + \epsilon$ in Fig. 1(b), along with results of quantal computations of the ensemble averaged energy.

We show that the resonant peaks are determined by ϵ -classical phase-space structures. It is convenient to change variables to $J = \pm I + \pi\ell + \tau\beta$, $\vartheta = \theta + \pi[1 - \text{sgn}(\epsilon)]/2$, thus turning the maps (5) into a single ϵ -classical standard map (ϵ SM), independent of the value of β :

$$J_{t+1} = J_t + \tilde{k}\sin(\vartheta_{t+1}), \quad \vartheta_{t+1} = \vartheta_t + J_t. \quad (6)$$

It turns out that (i) the resonant value $\tau = 2\pi\ell$ corresponds to integrable ϵ -classical dynamics, and the resonant values of the QM β correspond to the ϵ -classical resonant values of the action J ; (ii) for small $|\epsilon|$, the ϵ -classical dynamics is quasi-integrable, and the growth of the quantum energy is dominated by the main ϵ -classical resonant island around $J = 2\pi$; (iii) at any time t , the ratio between the energy and its value at $\epsilon = 0$ is a scaling function, and notably it is a function of the single variable $x = t\sqrt{k|\epsilon|}$ and not of t, k, ϵ separately.

We assume for simplicity an initially flat distribution of $p \in [0, 1]$; then $I_0 = 0$, and $J_0 = \pi\ell + \tau\beta_0$ with β_0 uniformly distributed in $[0, 1)$. Without loss of generality we also consider $\ell = 1$. Hence, if $|\epsilon| \ll 1$, then J_0 is uniformly distributed over one period (in action) $(\pi, 3\pi)$ of the ϵ SM. The resonant QM value is $\beta = 1/2$ [6]. Since $J_t = \pm I_t + \pi + \tau\beta$, and $I_0 = 0$, the mean energy of the rotor at time t is

$$\langle E_{t,\epsilon} \rangle = \epsilon^{-2} \langle I_t^2 \rangle / 2 = \epsilon^{-2} \langle (\delta J_t)^2 \rangle / 2, \quad \delta J_t = J_t - J_0.$$

The exact quantum resonance $\epsilon = 0$ corresponds to the integrable limit of the ϵ SM, where $\delta J_t = 0$. However, $\langle E_{t,\epsilon} \rangle$ is scaled by ϵ^{-2} , so in order to compute it at $\epsilon = 0$ one has to compute δJ_t at first order in ϵ . This leads to

$$\delta J_t = |\epsilon|k \sum_{s=0}^{t-1} \sin(\theta_0 + J_0 s) + r(\epsilon, t), \quad (7)$$

where $r(\epsilon, t) = o(\epsilon)$ as $\epsilon \rightarrow 0$ at fixed t . The particle energy at time t is found from (7) by taking squares, averaging over θ_0, J_0 , dividing by $2|\epsilon|^2$, and finally letting $\epsilon \rightarrow 0$:

$$\langle E_{t,0} \rangle = \frac{k^2}{8\pi} \int_{-\pi}^{\pi} dJ_0 \frac{\sin^2(J_0 t/2)}{\sin^2(J_0/2)} = \frac{k^2}{4} t. \quad (8)$$

The small contribution of the initial QM in the atom's energy was neglected. Apart from that, Eq. (8) is equal to the exact quantum mechanical result [6]. The integral over J_0 in (8) collects contributions from all the invariant tori $J_0 = \text{const}$ of the ϵ SM at $\epsilon = 0$, but it is essentially determined by a small interval $\sim 2\pi/t$ of actions around $J_0 = 2\pi$, the main ϵ -classical resonant torus. As that torus is formed of (period 1) fixed points, its own contribution is quadratic in time, so the linear growth (8) follows. Noting that $J_0 = 2\pi$ corresponds to the resonant QM, $\beta_0 = 1/2$, we see that the ϵ -quasi-classical approximation explains the *quantum* resonances in terms of *classical* resonances of the standard map [11]. We now

estimate $\langle E_{t,\epsilon} \rangle$ for $|\epsilon| > 0$. The $|\epsilon| > 0$ dynamics is maximally distorted with respect to the $\epsilon = 0$ one for J_0 in the vicinity of $2n\pi$, that is, in the very region which is mostly responsible for the linear growth of energy at $\epsilon = 0$. Being formed of period-1 fixed points, the $J_0 = 2\pi, \epsilon = 0$ invariant torus breaks at $|\epsilon| > 0$ as described by the Poincaré-Birkhoff theorem [11], giving rise to the “main resonance” of the ϵ SM, which is located astride $J = 2\pi$ with a size $\delta J_{\text{res}} \approx 4(k|\epsilon|)^{1/2}$ [11]. The approximation (7) fails quickly therein so its contribution $\langle E_{t,\epsilon} \rangle_{\text{res}}$ in the mean energy has to be estimated differently. In the remaining part of the ϵ -classical phase space the motion mostly follows Kolmogorov-Arnold-Moser invariant curves, slightly deformed with respect to the $\epsilon = 0$ ones, still with the same rotation angles. The contribution of such invariant curves in the mean energy is therefore roughly similar to that considered in the integral (8). On such grounds, in order to estimate $\langle E_{t,\epsilon} \rangle$ we remove from the integral (8) the contribution of the resonant action interval near $J_0 = 2\pi$, and we replace it by

$$\langle E_{t,\epsilon} \rangle \sim \frac{k^2}{4} t - \Phi(t) + \langle E_{t,\epsilon} \rangle_{\text{res}}, \quad (9)$$

where

$$\Phi(t) = \frac{k^2}{8\pi} \int_{-\delta J_{\text{res}}/2}^{\delta J_{\text{res}}/2} dJ' \frac{\sin^2(tJ'/2)}{\sin^2(J'/2)}, \quad (10)$$

and J' is the deviation from the resonant value 2π . $\langle E_{t,\epsilon} \rangle_{\text{res}}$ may be estimated by means of the pendulum approximation [11]. Near the ϵ SM resonance, the motion is described (in *continuous* time) by the pendulum Hamiltonian in the coordinates J', ϑ : $H_{\text{res}} = \frac{1}{2}(J')^2 + |\epsilon|k \cos(\vartheta)$. A characteristic time scale for the motion in the resonant zone is $t_{\text{res}} = (k|\epsilon|)^{-1/2}$, which is the period of the small pendulum oscillations divided by 2π . One may altogether remove $|\epsilon|$ from the Hamiltonian equations, by scaling momentum and time by factors $(k|\epsilon|)^{-1/2} = 4/\delta J_{\text{res}}$, $(k|\epsilon|)^{1/2} = 1/t_{\text{res}}$, respectively. Therefore,

$$\langle (\delta J_t)^2 \rangle = \langle (J'_t - J'_0)^2 \rangle \sim k|\epsilon| G(t\sqrt{k|\epsilon|}), \quad (11)$$

for an ensemble of orbits started inside the resonant zone, where $G(\dots)$ is a parameter-free function. This function results from averaging over nonlinear pendulum motions with a continuum of different periods, so it saturates to a constant value when the argument is larger than ≈ 1 . The contribution to the total energy is then obtained when multiplying (11) by $|\epsilon|^{-2} \delta J_{\text{res}} / (4\pi)$, because only a fraction $\sim \delta J_{\text{res}} / (2\pi)$ of the initial ensemble is trapped in the resonant zone. As a result,

$$\langle E_{t,\epsilon} \rangle_{\text{res}} \sim \pi^{-1} |\epsilon|^{-1/2} k^{3/2} G(t\sqrt{k|\epsilon|}). \quad (12)$$

When δJ_{res} is small, $\sin^2(J'/2)$ may be replaced by $J'^2/4$ in the integrand in (10), leading to

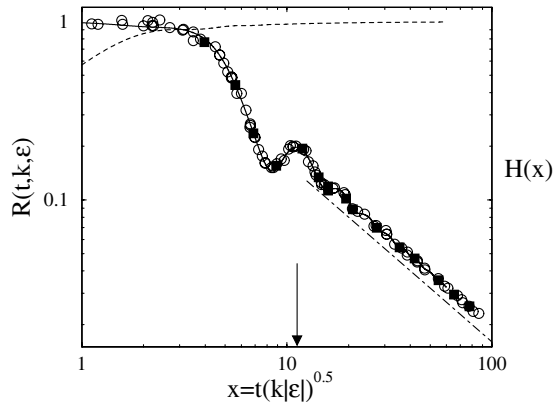


FIG. 2. Demonstrating the scaling (13) at $\tau \approx 2\pi$. Open circles correspond to different values of the parameters ϵ, k, t , randomly and independently generated in the ranges $1 < t < 200$, $0.001 < \epsilon < 0.1$, $0.1 < k < 50$, with the constraint $0.01 < k\epsilon < 0.2$. In each case an ensemble of 2×10^6 ϵ -classical rotors was used to numerically compute the scaled energy $R(t, k, \epsilon)$ (13), with uniform distributions of initial momenta in $[0, 1]$ and of initial θ in $[0, 2\pi)$. Full squares present quantum data for $k = 0.8\pi$, $t = 50$, and $t = 200$. The solid line is the scaling function $H(x)$ of (13) obtained by computation of the functions $\Phi_0(x)$ (dashed) and $G(x)$. The dot-dashed line has slope -1 . The arrow marks the value of the scaled detuning x corresponding to the arrow in Fig. 1(b).

$$\Phi(t) \sim \frac{k^2}{4} t \Phi_0(t\sqrt{k|\epsilon|}), \quad \Phi_0(x) \equiv \frac{2}{\pi} \int_0^x ds \frac{\sin^2(s)}{s^2}.$$

Replacing in (9), we obtain

$$R(t, k, \epsilon) \equiv \frac{\langle E_{t,\epsilon} \rangle}{\langle E_{t,0} \rangle} \sim H(x) \equiv 1 - \Phi_0(x) + \frac{4}{\pi x} G(x), \quad (13)$$

$$x = t\sqrt{k|\epsilon|} = t/t_{\text{res}}.$$

Hence $R(t, k, \epsilon)$ depends on t, k, ϵ only through the scaling variable $x = t/t_{\text{res}}$. The width in ϵ of the resonant peak therefore scales like $(kt^2)^{-1}$. The scaling law (13) is demonstrated by numerical data shown in Fig. 2. The function $H(x)$ was numerically computed: in particular, $G(x)$ was obtained by a standard Runge-Kutta integration of the pendulum dynamics. The scaling function $H(x)$ decays proportional to x^{-1} at large x , as do $1 - \Phi_0$ and $4G(x)/(\pi x)$, the latter owing to the saturation of G . Since Φ_0 is quite slowly varying at $x > 4$, the structures observed in that region are due to $G(x)$, which describes the resonant island.

Our analysis neglects higher-harmonics resonances of the ϵ SM, higher order islands, and the growth of the stochastic layer surrounding the main resonance [6]. It is therefore valid only if $k|\epsilon|$ is much smaller than 1, which is roughly the threshold for global chaos. In the case where the smooth initial momentum distribution includes values $n_0 \neq 0$ and/or is appreciably nonuniform in QM, the statistical weights of the various phase-space

regions are different. Scaling in the single variable t/t_{res} still holds, but the scaling functions Φ_0 and G may be different.

The map (5) is easily adapted to the model in the presence of decoherence due to random momentum jumps induced by external noise, e.g., the spontaneous emissions effects used in experiments [9,12]. A scaling law is again valid for the resonant peaks, in the *two* variables t/t_{res} and t_c/t_{res} where t_c is the time scale associated with noise [6].

To summarize, we have exploited a correspondence between the dynamics of a kicked particle near the quantum resonances $\tau = 2\pi\ell$ and the classical dynamics of a quasi-integrable system to analyze the structure of the experimentally observable quantum resonant peaks. For these we have derived a scaling law where the scaling variable is the ratio between the observation time and the characteristic time scale of elliptic motion inside the ϵ -classical island. This law provides significant new information about the shape and the parameter dependence of the peaks.

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