Obtaining the Apex of the Unitarity Triangle from $B \to \pi K$ Decays

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We present a method of obtaining the apex of the unitarity triangle from measurements of $B \to \pi K$ decay rates alone. Electroweak penguin amplitudes are included, and are related to tree operators. Discrete ambiguities are removed by comparing solutions with independent experimental data. The theoretical uncertainty in this method is about 10%.

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Within the standard model (SM), CP violation is due to a complex phase in the Cabibbo-Kobayashi-Maskawa (CKM) quark mixing matrix. It has become standard to parametrize this phase information using the unitarity triangle, whose apex is given by the CKM parameters (ρ, η), and in which the interior (CP-violating) angles are known as α, β, and γ [1]. One of the most important tasks in high-energy physics is to measure these quantities, and to test whether the SM explanation is correct.

Many methods have been proposed for getting α, β, and γ (or, equivalently, ρ and η). Most require the measurement of CP-violating asymmetries in hadronic B decays [2]. The most promising of these involve decays which are dominated by a single decay amplitude (e.g., $B_d^0(t) \to J/\psi K_S$, ϕK_S). However, many decays receive both tree and penguin contributions, with different weak phases, thus spoiling the cleanliness of the methods [3].

A number of years ago, it was shown that an isospin analysis of $B \to \pi\pi$ decays allows one to remove the penguin "pollution" in $B_d^0(t) \to \pi^+\pi^-$, so that the CP phase α can be measured [4]. Subsequently, Nir and Quinn (NQ) showed that this technique could also be applied to $B \to \pi K$ decays, giving another way of extracting α [5]. Unfortunately, this analysis neglects electroweak penguin operators (EWP's), and such operators are very important in $B \to \pi K$ decays [6]. When one includes EWP's, the NQ $B \to \pi K$ analysis fails—one cannot obtain weak-phase information. [The validity of analyses which rely on SU(3) relations between $B \to \pi K$ and $B \to \pi\pi$ decays [7,8] is also affected by EWP's.]

Recently, it was shown that, by using Fierz transformations and SU(3) symmetry, it is possible to relate EWP's to tree operators [9,10]. In light of this, the $B \to \pi K$ analysis can be resuscitated and improved. As we will show, the apex of the unitarity triangle can be obtained from measurements of the $B \to \pi K$ decay rates alone. In general, the discrete ambiguities can be removed by comparison with experimental data. The theoretical error in this method is rather small, about 10%.

Using isospin, the $B \to \pi K$ amplitudes satisfy a quadrilateral relation:

$$A^{+0} + \sqrt{2}A^{0+} = \sqrt{2}A^{00} + A^{-+}, \tag{1}$$

where we have defined $A^{ij} \equiv A(B \to \pi^i K^j)$. The CP-conjugate amplitudes $\bar{A}^{ij}$ satisfy a similar relation (note, $\bar{A}^{+0}$ corresponds to $B^- \to \pi^- \bar{K}^0$, etc.). It is possible to express all amplitudes in terms of a number of distinct operators. This is equivalent to a description in terms of diagrams [7]. Neglecting the exchange- and annihilation-type diagrams, which are expected to be small for dynamical reasons, but including EWP's, there are five diagrams which contribute to $B \to \pi K$ decays [8,11]: (i) a color-favored tree amplitude T; (ii) a color-suppressed tree amplitude C; (iii) a gluonic penguin amplitude P; (iv) a color-favored electroweak penguin amplitude P_{EW}; (v) a color-suppressed electroweak penguin amplitude P_{EW}^C [12]. The $B \to \pi K$ amplitudes can then be written [11]

$$\begin{aligned} A^{+0} &= P - \tfrac{1}{3}P_{EW}^C, \\ \sqrt{2}A^{0+} &= -P - Te^{i\gamma} - Ce^{i\gamma} - P_{EW} - \tfrac{2}{3}P_{EW}^C, \\ \sqrt{2}A^{00} &= P - Ce^{i\gamma} - P_{EW} - \tfrac{1}{3}P_{EW}^C, \\ A^{-+} &= -P - Te^{i\gamma} - \tfrac{2}{3}P_{EW}^C. \end{aligned} \tag{2}$$

Here we have explicitly written the weak phase (γ), while P, T, etc. implicitly include strong phases. To obtain the amplitudes $\bar{A}^{ij}$ for the CP-conjugate processes, one simply changes the sign of the weak phases. We have assumed that the $b \to s$ penguin contribution P and the EWP's are dominated by the internal t quark, so that they have no weak phase in the Wolfenstein parametrization [1]. In this case, the amplitudes A^{+0} and $\bar{A}^{+0}$ are identical.

In the above, the (complex) $B \to \pi K$ amplitudes are written in terms of the six quantities P, T, C, P_{EW}, P_{EW}^C, and $e^{i\gamma}$. First, suppose that EWP's are absent. In this case seven $B \to \pi K$ branching ratios can be measured, but one has only six theoretical quantities: $|P|$, $|T|$, $|C|$, two relative strong phases, and γ. Thus, one can solve for the theoretical parameters. The first step is to invert the expressions for the amplitudes in order to write the theoretical quantities in terms of the magnitudes and relative phases of four of the A^{ij} and $\bar{A}^{ij}$. Now, we can get the magnitudes of A^{ij} and $\bar{A}^{ij}$ from measurements of the $B \to \pi K$ branching ratios. However, in order to obtain the relative phases, we must fix the A quadrilateral and the $\bar{A}$ quadrilateral, and know their relative orientations. In order to do this, we need two additional (real) relations involving the A^{ij} and $\bar{A}^{ij}$. In the absence of EWP's, such relations exist. They are

$$A^{-+} + \sqrt{2}A^{00} = \tilde{A}^{-+} + \sqrt{2}\tilde{A}^{00},$$
$$\sqrt{2}A^{00} + \sqrt{2}A^{0+} = \sqrt{2}\tilde{A}^{00} + \sqrt{2}\tilde{A}^{0+}, \tag{3}$$

where $\tilde{A}^{ij} \equiv e^{2i\gamma}\bar{A}^{ij}$. (A third relation, not independent, is $A^{+0} + A^{-+} = \tilde{A}^{+0} + \tilde{A}^{-+}$.) The first equation above indicates that the A and $\tilde{A}$ quadrilaterals share a common diagonal, the isospin-3/2 amplitude:

$$A_{3/2} = A^{-+} + \sqrt{2}A^{00} = -(T + C)e^{i\gamma}. \tag{4}$$

Obviously, since the diagonals are common to both quadrilaterals, they have the same length. The second relation is used to determine this length. With this knowledge, we can fix the A and the $\bar{A}$ quadrilaterals, and determine their relative orientations. Thus, in the absence of EWP's, it is possible to solve for the six theoretical quantities, including the weak phase γ. This is the NQ method [5].

Unfortunately, in the presence of EWP's, it is no longer possible to do this [6,11]. In this case there are ten theoretical quantities but only seven experimental measurements. Thus, it is impossible to express the theoretical quantities in terms of the A^{ij} and $\bar{A}^{ij}$. [It is also straightforward to verify that Eqs. (3) above no longer hold if P_{EW} and P_{EW}^C are nonzero.] It therefore appears impossible to obtain weak-phase information from $B \to \pi K$ decays.

Fortunately, to a good approximation, the EWP's are not independent quantities—P_{EW} and P_{EW}^C can be related to T and C. Briefly, the argument goes as follows [9,10]. The SM effective weak Hamiltonian for $B \to \pi K$ decays is

$$H_{\rm eff}^q = \frac{G_F}{\sqrt{2}}\bigg[V_{ub}V_{us}^*(c_1O_1 + c_2O_2) - \sum_{i=3}^{10} V_{tb}V_{ts}^* c_i O_i\bigg] + {\rm H.c.} \tag{5}$$

In the above, O_1 and O_2 are $(V - A) \times (V - A)$ tree operators, while O_7–O_{10} describe the electroweak penguin operators. O_7 and O_8 have the Lorentz structure $(V - A) \times (V + A)$, while O_9 and O_{10} are $(V - A) \times (V - A)$. However, the Wilson coefficients c_7 and c_8, which multiply O_7 and O_8, are much smaller than c_9 and c_{10} [13]:

$$c_7 = 3.49 \times 10^{-4}, \qquad c_8 = 3.72 \times 10^{-4},$$
$$c_9 = -9.92 \times 10^{-3}, \qquad c_{10} = 2.54 \times 10^{-3}. \tag{6}$$

Thus, the EWP's are approximately given purely by O_9 and O_{10}. Furthermore, these operators can be Fierz transformed into O_1 and O_2, since all have a $(V - A) \times (V - A)$ structure. Therefore the EWP's are related to the tree operators.

There are two independent relations between EWP's and tree operators. Ignoring exchange- and annihilation-type diagrams once again, they are given by [10]

$$P^{EW}(B^+ \to \pi^+K^0) + \sqrt{2}P^{EW}(B^+ \to \pi^0K^+) = -\frac{3}{2}\frac{c_9 + c_{10}}{c_1 + c_2}\frac{(T + C)}{|V_{ub}^*V_{us}|},$$
$$P^{EW}(B^0 \to \pi^-K^+) + P^{EW}(B^+ \to \pi^+K^0) = \frac{3}{4}\frac{c_9 - c_{10}}{c_1 - c_2}\frac{(T - C)}{|V_{ub}^*V_{us}|} - \frac{3}{4}\frac{c_9 + c_{10}}{c_1 + c_2}\frac{(T + C)}{|V_{ub}^*V_{us}|}. \tag{7}$$

Using the expressions for the $B \to \pi K$ amplitudes given in Eq. (2), these give

$$P_{EW} = \frac{3}{4}\frac{c_9 + c_{10}}{c_1 + c_2}R(T + C) + \frac{3}{4}\frac{c_9 - c_{10}}{c_1 - c_2}R(T - C), \qquad P_{EW}^C = \frac{3}{4}\frac{c_9 + c_{10}}{c_1 + c_2}R(T + C) - \frac{3}{4}\frac{c_9 - c_{10}}{c_1 - c_2}R(T - C), \tag{8}$$

where $R \equiv |V_{tb}^*V_{ts}/V_{ub}^*V_{us}|$. These provide the relations between the diagrams P_{EW} and P_{EW}^C and the tree operators T and C.

In Refs. [9,10], a numerical value is taken for R. In our method, this is not necessary. Instead, we keep the general expression

$$\left|\frac{V_{tb}^*V_{ts}}{V_{ub}^*V_{us}}\right| = \frac{1}{\lambda^2\sqrt{\rho^2 + \eta^2}}. \tag{9}$$

As we will see below, this allows us to improve considerably upon the original NQ method.

With the relations in Eq. (8), the various $B \to \pi K$ decays are described by seven theoretical parameters: $|P|$, $|T|$, $|C|$, two relative strong phases, and *two* pieces of weak-phase information, which we take to be γ and $\sqrt{\rho^2 + \eta^2}$. However, there are seven experimental measurements of $B \to \pi K$ branching ratios, so that we can solve for all theoretical parameters. Note that the knowledge of γ and $\sqrt{\rho^2 + \eta^2}$ is sufficient to pin down the shape of the unitarity triangle. Thus, one can obtain the full unitarity triangle (up to discrete ambiguities) from measurements of the $B \to \pi K$ rates alone.

Note that, in light of the relations between EWP's and tree operators, the $A_{3/2}$ amplitude in Eq. (4) is given by

$$A_{3/2} = -(T + C)\left[e^{i\gamma} + \frac{3}{2}\frac{c_9 + c_{10}}{c_1 + c_2} R \right]. \quad (10)$$

This is one of the diagonals of the A quadrilateral. The corresponding diagonal in the $\bar{A}$ quadrilateral is given by the above expression, but with $\gamma \to -\gamma$. The relations in Eq. (11) imply that $|A_{3/2}| = |\bar{A}_{3/2}|$, as was true for the case without EWP's. The magnitudes and relative phases of the $B \to \pi K$ amplitudes are therefore obtained by measurements of the branching ratios and the construction of the A and $\bar{A}$ quadrilaterals. This allows us to obtain all the theoretical parameters.

Analytically, this is done as follows. The $B \to \pi K$ amplitudes are written in terms of the five (complex) theoretical quantities P, T, C, $e^{i\gamma}$, and $\sqrt{\rho^2 + \eta^2}$. One can invert these expressions to write the theoretical parameters in terms of five independent A^{ij} and $\bar{A}^{ij}$ amplitudes [14]. However, as discussed earlier, in order to determine these parameters, one needs two additional (real) relations to fix the two quadrilaterals and their relative orientation. There are several ways to obtain these. One is to note that, at this stage, $e^{i\gamma}$ and $\sqrt{\rho^2 + \eta^2}$ are simply arbitrary complex quantities, and are expressed in terms of the A^{ij} and $\bar{A}^{ij}$. However, there are physical constraints on these parameters. They are

$$|e^{i\gamma}| = 1, \qquad \mathrm{Im}(\sqrt{\rho^2 + \eta^2}) = 0. \quad (11)$$

These provide the relations necessary to fix the relative orientations of the two quadrilaterals.

We now demonstrate numerically how the method works. Ideally, we would use current experimental data on $B \to \pi K$ rates. Unfortunately, although the various branching ratios have been measured, no significant partial-rate asymmetries have yet been observed [1], and our method requires at least one measurement of direct CP violation. We therefore generate values for the "experimental measurements" by assuming input values for P, T, etc. We choose

$$|P| = 1.0, \quad \delta_P = -18.0°, \quad |T| = 0.3, \quad \delta_T = 2.0°, \quad |C| = 0.05, \quad \delta_C = 102.0°, \quad \rho = 0.18, \quad \eta = 0.38. \quad (12)$$

With these inputs, we find

$$|A^{+0}| = |\bar{A}^{+0}| = 1.00, \quad |A^{0+}| = 0.86, \quad |\bar{A}^{0+}| = 1.00, \quad |A^{00}| = 0.62, \quad |\bar{A}^{00}| = 0.57, \quad |A^{-+}| = 1.07, \quad |\bar{A}^{-+}| = 1.22. \quad (13)$$

Here we have taken the values for c_9 and c_{10} given in Eq. (6), along with $c_1 = 1.144$ and $c_2 = -0.308$ [13].

Given this "experimental data," we can solve the system for our seven theoretical unknowns. Since the equations are nonlinear, there will be many discretely ambiguous solutions. For the "data" in Eq. (13), we find 16 solutions. Half of these yield unitarity triangles which point down, i.e., $\eta < 0$. However, we know from the kaon system that $\eta > 0$ [15]. We therefore exclude solutions with $\eta < 0$. The eight remaining solutions are shown in Table I. [Note: We have solved the system for many different inputs [Eq. (12)]. In all cases, we find either 16 or eight solutions, half of which can be rejected because $\eta < 0$.]

Now, if the SM is correct, there are several constraints which these putative solutions must satisfy. First, CP violation in $B_d^0(t) \to J/\psi K_S$ has been measured, yielding a world average of $\sin 2\beta = 0.736 \pm 0.049$ [16]. Any solution in Table I which does not give a value for $\sin 2\beta$ in its 3σ range is excluded. Second, the latest 95% C.L. range for $S \equiv \sqrt{\rho^2 + \eta^2}$ is $0.356 \le S \le 0.452$ [17]. An acceptable solution must give a value for S in this range. Finally, some solutions can be eliminated by making the mild theoretical assumption that $|P| > |T| > |C|$. (This constraint is not essential—we find that solutions which do not satisfy this condition generally also violate one of the experimental constraints.)

In the particular case of Table I, the experimental constraints alone eliminate all solutions except (3) (the true solution). Indeed, in almost all of the cases we studied, in which we varied the inputs in Eq. (12), we found that only a single solution remained after imposing the constraints. Thus, it is in fact possible to obtain the apex of the unitarity triangle from measurements of the $B \to \pi K$ rates alone.

TABLE I. The eight sets of theoretical parameters which reproduce the "experimental data" of Eq. (13). We also give the predicted values of each set for $\sin 2\beta$, $S \equiv \sqrt{\rho^2 + \eta^2}$, and $A_{\pi K}^{\mathrm{indir}}$, the indirect CP asymmetry in $B_d^0(t) \to \pi^0 K_S$.

	$\|P\|$	$\|T\|$	$\|C\|$	$(\delta_P - \delta_T)$	$(\delta_T - \delta_C)$	ρ	η	$\sin 2\beta$	S	$A_{\pi K}^{\mathrm{indir}}$
(1)	0.96	0.42	0.33	−126.5°	−28.0°	−0.56	0.17	0.21	0.59	−0.41
(2)	0.98	1.97	1.74	−21.0°	−149.2°	−7.33	0.99	0.23	7.40	0.31
(3)	1.0	0.3	0.05	−20.0°	−100.0°	0.18	0.38	0.76	0.42	−0.80
(4)	1.01	1.90	0.43	−31.9°	−55.0°	−1.91	0.18	0.12	1.91	−0.09
(5)	1.02	1.70	0.55	−46.7°	−26.7°	−0.96	0.07	0.07	0.96	−0.03
(6)	1.02	1.60	0.23	−4.6°	−8.6°	−0.88	0.91	0.78	1.27	−0.48
(7)	1.08	2.05	0.59	−5.1°	−2.4°	−0.68	0.37	0.42	0.77	0.24
(8)	1.38	3.12	1.18	−5.5°	−0.6°	−0.28	0.06	0.10	0.29	−0.67

Note that it is also possible to measure independently the indirect CP asymmetry in $B_d^0(t) \to \pi^0 K_S$:

$$A_{\pi K}^{\rm indir} \equiv \frac{{\rm Im}(e^{-2i\beta} A^{00*} \bar{A}^{00})}{|A^{00}||\bar{A}^{00}|}. \quad (14)$$

The knowledge of this quantity will provide a cross-check to the solution(s) found above. If two solutions happen to be found, then one can in principle distinguish between them through the measurement of $A_{\pi K}^{\rm indir}$. If one finds only a single solution using the above method, $A_{\pi K}^{\rm indir}$ furnishes an independent check of this solution.

It is possible that no solution is found which satisfies all the constraints and independent measurements. This would then be evidence for physics beyond the SM. Indeed, the present measurement of the indirect CP asymmetry in $B_d^0(t) \to \phi K_S$ may be showing signs of new physics: Although the BaBar measurement is in agreement with the SM prediction (within errors), the Belle measurement disagrees at the level of 3.5σ [16]. If this discrepancy with the SM is confirmed, this would point specifically to new physics in the $b \to s$ penguin amplitude [18]. Since $B \to \pi K$ decays also involve $b \to s$ penguin diagrams, they would also be affected by this new physics. In particular, we would expect to find a discrepancy in the values of the parameters of the unitarity triangle as extracted using the above method and in other, independent measurements (e.g., $\sin 2\beta$).

There are several sources of theoretical input in this method. First, we ignore annihilation- and exchange-type diagrams. In Refs. [7,11], these are estimated to be about 1% of the dominant decay amplitude, though this must be verified experimentally through measurements of decays such as $B_d^0 \to D_s^+ D_s^-$. Second, we have assumed that both the gluonic and electroweak penguin diagrams are dominated by the internal t quark. The error here is at the level of $|V_{ub}^* V_{us}/V_{tb}^* V_{ts}| \simeq 2\%$. Third, we neglect the Wilson coefficients c_7 and c_8 compared to c_9 and c_{10}, giving an error of about 4% [see Eq. (6)]. Finally, one must take SU(3)-breaking effects into account in Eq. (8). Reference [9] estimates such effects, and finds them to be roughly 5%, though they may be larger [19]. Thus, the net theoretical error in this method is about 10%.

Finally, we note that experimental errors in the $B \to \pi K$ branching ratios may make it challenging to implement this method in practice. Indeed, we find that these must be measured with a precision at the percent level in order to distinguish among the discrete ambiguities of Table I. However, it should be possible to improve upon this by performing a simultaneous fit to all experimental data, including constraints from $\sin 2\beta$ and $\sqrt{\rho^2 + \eta^2}$.

In summary, we have presented a method of obtaining the apex of the unitarity triangle from measurements of $B \to \pi K$ rates alone. It relies on a relation between electroweak penguin amplitudes and tree operators. One can distinguish among discretely ambiguous solutions by using independent experimental determinations of $\sin 2\beta$ and $\sqrt{\rho^2 + \eta^2}$. The theoretical uncertainty is $\sim$10%. Although B factories have measured the $B \to \pi K$ rates, no difference between the B and $\bar{B}$ branching ratios has been observed. As soon as one observation of direct CP violation in $B \to \pi K$ decays is made, it should be possible to extract the full unitarity triangle using this method.

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