Modulational and Filamentational Instabilities of Intense Photon Pulses and Their Dynamics in a Photon Gas

P. K. Shukla and B. Eliasson[*]

Institut für Theoretische Physik IV, Ruhr-Universität Bochum, D-44780 Bochum, Germany
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It is shown that an intense photon pulse interacting nonlinearly with sound waves in a photon gas is subjected to modulational and filamentational instabilities. Starting from a new set of coupled equations governing nonlinear photon-photon interactions, we derive a dispersion relation which depicts the temporal and spatial amplification rates of the modulational and filamentational instabilities. The long term behavior of the modulationally unstable waves renders collapse of a photon beam as well as the formation of cylindrically symmetric photonic solitons. The results can have relevance to the understanding of the nonlinear photonic pulse propagation in astrophysical environments as well as in forthcoming intense laser-matter interaction experiments.

In his classic paper, Karpman [1] derived a pair of equations (hereafter referred to as the Karpman equations) which describe the dynamics of nonlinear coupled high-frequency electromagnetic and ion-acoustic waves in a nonrelativistic plasma. The Karpman equations are useful for studying the modulational-filamentational instability [2] of large amplitude electromagnetic waves in an unmagnetized plasma. Bingham *et al.* [3] considered the nonlinear coupling between incoherent photons (quasiparticles) and plasmons and reported the damping or growth of Langmuir waves due to quasiparticle-wave interactions. The nonlinear interaction is governed by the Liouville equation [4–6] for random-phased quasiparticles (photons) and a driven (by the quasiparticle ponderomotive force) Langmuir wave equation. Depending upon the quasiparticle electric field spectrum, one can have both damping and growth of Langmuir waves in plasmas.

However, when the electromagnetic wave intensity is extremely high, there appear new nonlinear interactions, due to quantum electrodynamics effects (QED) [7–9], between an electromagnetic pulse and a radiation background composed of ultrarelativistic particles [10]. Specifically, photon-photon scattering in vacuum is one of the most fundamental mechanisms which can give rise to the nonlinear optical effects. The expected nonlinear interactions come from virtual creation and annihilation of electron-positron pairs, giving rise to self-coupling of strong electromagnetic fields via the vacuum polarization. In the remote past, Bialynicka-Birula and Bialynicki-Birula [11] considered photon propagation and photon splitting in an external field and pointed out the importance of nonlinear effects predicted by the QED model. The photon-photon scattering of intense laser radiation predicted by QED can give rise to second harmonic generation in a dc magnetic field due to broken symmetry of the interaction. Nonlinear magneto-optics of vacuum has been presented by Ding and Kaplan [12]. Four wave interactions of intense radiation in an electron-positron vacuum have been analyzed by Rozanov [13], while the effects of homogeneous electric and magnetic fields on self-action of intense coherent electromagnetic waves have been studied in Ref. [14], including diffraction, spatiotemporal dispersion, and vacuum nonlinearity.

Brodin *et al.* [15] proposed detection of QED vacuum nonlinearities in Maxwell's equations by the use of waveguides. They showed that photon-photon scattering gives rise to self-interaction terms which are similar to the nonlinearities due to the polarization in nonlinear optics [16]. Based on their new nonlinear terms, they reported the excitation of new modes which should facilitate detection of QED associated nonlinearities in waveguides. The self-interaction terms can also cause optical collapse in vacuum [17]. In a recent paper, Marklund *et al.* [18] have investigated the nonlinear interaction, due to QED effects, between an intense electromagnetic pulse and a radiation background. They derived a pair of equations, which we refer to as the Marklund-Brodin-Stenflo equations, governing the nonlinear coupling between modulated photon pulses and driven (by the radiation pressure) acousticlike perturbations in the photon gas. They briefly discussed focusing and collapse of an intense short photon pulse.

In this Letter, we use the Marklund-Brodin-Stenflo equations [18] to investigate the modulational and filamentational instabilities of an intense short electromagnetic pulse propagating through a photon gas. For this purpose, we derive a nonlinear dispersion relation and analyze it both analytically and numerically. We also present a simulation study of spatiotemporal evolution of modulationally unstable photon wave packets as well as their localization in the form of a photon bullet. The present results should help to understand the nonlinear propagation of intense photons (e.g., gamma ray bursts) through the cosmic microwave background, as well as through an ionized medium that is produced owing to

intense laser-plasma interactions involved in the next generation inertial confinement fusion schemes.

Let us consider the propagation of a plane intense electromagnetic wave propagating through a radiation gas which supports acousticlike perturbations. The latter modulate the intense electromagnetic pulse whose dispersion properties are governed by [18]

$$\omega^2 = k^2 c^2 \left(1 - \tfrac{2}{3}\lambda\mathcal{E}\right), \tag{1}$$

where ω and $\mathbf{k}$ are the frequency and wave vector of intense photons, c is the speed of light in vacuum, and the energy density of the radiation gas is denoted by $\mathcal{E} = \mathcal{E}_0 + \mathcal{E}_1$, where the perturbed energy density of the radiation gas $\mathcal{E}_1 \ll$ the equilibrium value $\mathcal{E}_0$ in the absence of the electromagnetic pulse. By introducing the eikonal representation and the WKB approximation [19] (viz. $|\partial E_p/\partial t| \ll |\omega E_p|$ and $|\hat{\mathbf{k}}_p \cdot \nabla E_p| \ll |k_p E_p|$), we obtain from (1) a nonlinear Schrödinger equation for modulated photons [18]. We have

$$i\left(\frac{\partial}{\partial t} + c\hat{\mathbf{k}}_p \cdot \nabla\right)E_p + \frac{c}{2k_p}\nabla_\perp^2 E_p + \frac{2}{3}\lambda c k_p \mathcal{E}_1 E_p = 0, \tag{2}$$

where the electric and magnetic fields of the intense pulse are $\mathbf{E} = E_p \hat{\mathbf{e}}$ and $\mathbf{B} = E_p \hat{\mathbf{k}}_p \times \hat{\mathbf{e}}/c$, respectively, and $\hat{\mathbf{e}}$ is the unit vector. Furthermore, we have denoted $\nabla_\perp^2 = \nabla^2 - (\hat{k}_p \cdot \nabla)^2$ and $\lambda = \lambda_\pm$, where $\lambda_+ = 14\kappa$ and $\lambda_- = 8\kappa$ for the two different polarization states of the photons, and $\kappa = 2\alpha^2\hbar^3/45m_e^4c^5 \approx 1.63 \times 10^{-30}$ ms^2/kg; α is the fine constant, $\hbar$ is the Planck constant, and m_e is the electron mass.

A photon gas, which supports acousticlike perturbations, interacts with an intense photon pulse through self-interaction between photons. The directions $\hat{\mathbf{k}}_{ph} \equiv \mathbf{k}_{ph}/k_{ph}$ of the photons in the gas are approximately random, giving the effective acoustic speed $c/\sqrt{3}$ in the averaged fluid description of the photon gas. Subsequently, acousticlike perturbations in the photon bath are reinforced by the radiation pressure, and the dynamics of acousticlike perturbations is governed by [18]

$$\frac{\partial^2 \mathcal{E}_1}{\partial t^2} - \frac{c^2}{3}\nabla^2\mathcal{E}_1 = -\frac{2\lambda\epsilon_0\mathcal{E}_0}{3}\left(c^2\nabla^2|E_p|^2 + \frac{\partial^2|E_p|^2}{\partial t^2}\right), \tag{3}$$

where ϵ_0 is the permittivity of free space. Equation (3) has been derived by combining the radiation hydrodynamic equations (the momentum and energy conservation equations including the radiation pressure) in the photon gas. The pair of equations (2) and (3) are referred to as the Marklund-Brodin-Stenflo equations. They are significantly different from the Karpman equations [1].

The dispersion relation for the modulational and filamentational instabilities of a constant amplitude photon pump $(\omega_0, \mathbf{k}_0)$ can be found by linearizing Eqs. (2) and (3) around the unperturbed state $E_p = E_{p0}$ and $\mathcal{E}_1 = 0$. We consider perturbations of the form $E_p = E_{p0} + \hat{E}_{p1}^+ \exp(i\xi) + \hat{E}_{p1}^- \exp(-i\xi)$ and $\mathcal{E}_1 = \hat{\mathcal{E}}_1 \exp(i\xi)$, where E_{p0}, $\hat{E}^+$, and $\hat{E}^-$ are constants, and $\xi = \mathbf{K}\cdot\mathbf{r} - \Omega t$. Here Ω and $\mathbf{K}$ are the frequency and wave vector of the acousticlike perturbations. The standard procedure of the parametric instability analysis [20,21] can then be introduced on Eqs. (1) and (2), yielding the nonlinear dispersion relation

$$\left[(\Omega - c\hat{\mathbf{k}}_p \cdot \mathbf{K})^2 - \frac{K_\perp^4 c^2}{4k_p^2}\right](3\Omega^2 - K^2c^2) = \frac{4K_\perp^2 c^2}{3}(\Omega^2 + K^2c^2)\frac{\epsilon_0|E_{p0}|^2}{W}, \tag{4}$$

where $K^2 = K_x^2 + K_y^2 + K_z^2 \equiv K_x^2 + K_\perp^2$ and $W = 1/\lambda^2\mathcal{E}_0$ represents the energy density of the photon gas. The dispersion relation for the filamentational instability follows from (4) in the quasistationary limit, $\Omega = 0$, and for $K_x \ll K_\perp$, so that we have

$$K_x^2 = \frac{K_\perp^4}{4k_p^2} - \frac{4}{3}K_\perp^2\frac{\epsilon_0|E_{p0}|^2}{W}, \tag{5}$$

where $\hat{\mathbf{k}}_p \cdot \mathbf{K} = K_x$. Equation (5) depicts that the filamentation $(K_x = iK_i)$ of an intense photon beam sets in if $\epsilon_0|E_{p0}|^2 > 3K_\perp^2 W/16k_p^2$. The maximum spatial amplification rate, occurring at $K_\perp = \sqrt{8\epsilon_0/3}k_p|E_{p0}|/\sqrt{W}$, is $K_i = 4\epsilon_0|E_{p0}|^2/3W$.

Letting $\Omega = \Omega_r + i\Omega_i$ into Eq. (4) we solve it numerically for the real and imaginary parts (Ω_r and Ω_i, respectively) of the modulation frequency, and plot them against K/k_p in Figs. 1 and 2 for different values of k_x and the pump strength $A = \epsilon_0|E_{p0}|^2/W$, respectively. An intense photon beam with an electric field $E_{p0} \approx$

10^{34} V/m, propagating through a photon gas having the energy density $\mathcal{E}_0 \approx 0.1$ J/m^3 would give $A = \epsilon_0|E_{p0}|^2/W \approx 0.03$, which ensures the threshold criteria for the onset of modulational and filamentational instabilities. Figure 1 shows that the growth rate of the modulational instability is larger for higher values of $K_x/K_\perp$, and it attains a maximum value at certain $K_\perp/k_p$ values depending on the $K_x/K_\perp$ values. In Fig. 2 we see that the growth rate is higher for larger values of the photon pump intensity; the maximum growth rate scales linearly with A.

Second, in the steady state Eqs. (2) and (3) can be put in the form

$$2i\frac{\partial E_p}{\partial x} + \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial E_p}{\partial r}\right) + |E_p|^2 E_p = 0, \tag{6}$$

where $\sqrt{\epsilon_0}E_p$ is normalized by $3\sqrt{W}/2\sqrt{2}$ and the space variables x and r are in units of k_p^{-1}. Equation (6) is a
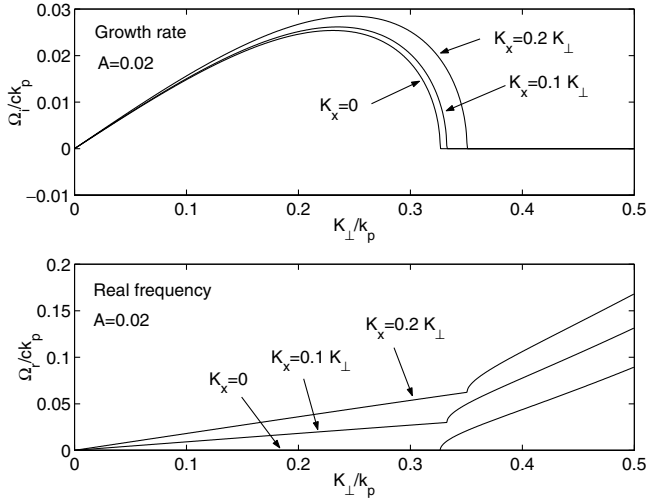
FIG. 1. The growth rate γ (upper panel) and real frequency Ω_{Re} (lower panel) as a function of the wave number $K_\perp$ for pump strength $A = 0.02$, where $A = \epsilon_0 |E_{p0}|^2/W$, and different values on K_x.

multidimensional cubic Schrödinger equation, which has the following integrals of motions [1]: $I_1 = \int_0^\infty r|E_p|^2 dr$ and $I_2 = \int_0^\infty (|\partial E_p/\partial r|^2 - |E_p|^4/2)r dr$, yielding the relation $I_2 = (\partial^2/\partial x^2)\int_0^\infty r^3|E_p|^2 dr$. For $I_2 < 0$ we have spatial collapse of the photon beam radius. The photon beam will have a steady state profile if $I_2 = 0$. In this case, the diffraction is exactly balanced by the nonlinear focusing effect, and one encounters a cylindrical soliton which depends on x through a complex phase, i.e., $E_p = F(r)\exp(ix/2)$, where R is real satisfying the cylindrical soliton equation

$$\frac{\partial^2 F}{\partial r^2} + \frac{1}{r}\frac{\partial F}{\partial r} + F^3 - F = 0. \qquad (7)$$
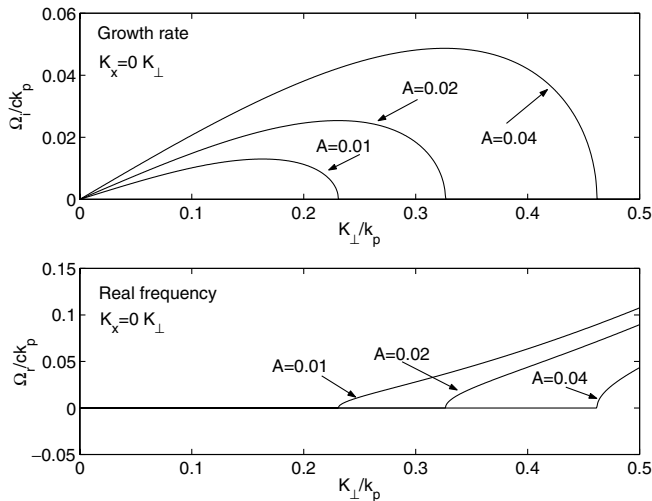
FIG. 2. The growth rate γ (upper panel) and real frequency Ω_{Re} (lower panel) as a function of the wave number $K_\perp$ for different pump strengths A, where $K_x = 0 K_\perp$.

Equation (7) has been numerically solved subjected to the boundary conditions that $\partial F/\partial r = 0$ and $F = F_0 \neq 0$ at $r = 0$ and $F(r) \propto K_0(r)$ for large r, where $K_0(r)$ is the modified Bessel function of the second kind. Since in general, for large r, $F(r)$ is a linear combination of $I_0(r)$ and $K_0(r)$, $F(0)$ has to be adjusted so as to eliminate the divergent solution $I_0(r)$, exactly as in any eigenvalue problem. It turns out that $F(r)$ is a monotonic decreasing function of r and approaches zero at large distances. This represents a cylindrically symmetric photon bullet, also known in nonlinear optics [16,22,23].

The coupled equations (2) and (3) are solved numerically in order to understand the spatiotemporal evolution of a multidimensional photonic pulse. The results are displayed in Fig. 3. The photon beam is propagating in the x direction in an initially homogeneous photon gas, giving rise to perturbations in the (y,z) plane; thus we assume that $\partial/\partial x$ vanishes and consider the purely two-dimensional problem in the (y,z) plane, plus time. The spatial y and z variables are scaled by k_p^{-1}, the time t by $(k_p c)^{-1}$, the energy fluctuations $\mathcal{E}_1$ by λ^{-1}, and the electric field E_p by $(\lambda\sqrt{\mathcal{E}_0\epsilon_0})^{-1}$. The resulting dimensionless and parameter-free system of equations is written in the form

$$\frac{\partial E_p}{\partial t} = i\left(\frac{1}{2}\nabla_\perp^2 E_p + \frac{2}{3}\mathcal{E}_1 E_p\right), \qquad (8)$$

$$\frac{\partial G}{\partial t} = \frac{1}{3}\nabla^2 \mathcal{E}_1 - \frac{2}{3}\nabla^2|E_p|^2, \qquad (9)$$

and $\partial F/\partial t = G$, where $F = \mathcal{E}_1 + (2/3)|E_p|^2$ and in the two-dimensional case studied here $\nabla_\perp^2 = \nabla^2 = \partial^2/\partial y^2 + \partial^2/\partial z^2$. We used periodic boundary conditions
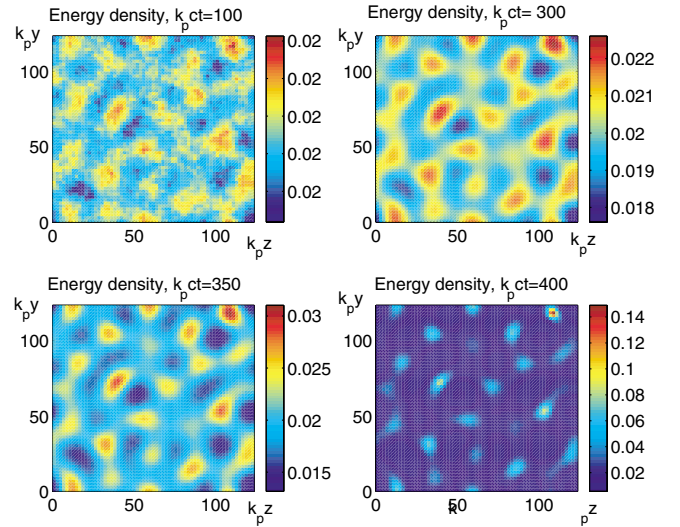
FIG. 3 (color online). The normalized photon energy density $\epsilon_0|E_p|^2/W$ as a function of the normalized spatial variables $k_p y$ and $k_p z$ at different times $k_p ct$. We see that initially uniformly distributed photon energy forms localized packages which self-focus and collapse.

in the domain $0 \leq y \leq 20\pi$ and $0 \leq z \leq 20\pi$, with 64 intervals in each direction. The y and z derivatives were approximated with a pseudospectral method and the system was advanced in time by the fourth-order Runge-Kutta scheme, with time step $\Delta t = 0.5$. The initial condition at time $t = 0$ was $E_p = \sqrt{0.02}$, corresponding to the case $A = 0.02$ in the theoretical investigation of Fig. 2. A low amplitude noise (random numbers) of the order 10^{-4} was added to E_p to give a seed for any instabilities.

The numerical results are displayed in Fig. 3 (in dimensional units), where we observe the self-focusing and collapse of wave packets occurring at different times. As can be seen in Fig. 3, the initially almost homogeneous electric field has at the time $k_p ct = 100$ developed wave packets, which at the last time $k_p ct = 400$ have collapsed to strongly localized wave packets. As predicted by Marklund et al. [18], we find that the energy and electric field scale approximately as $\mathcal{E}_1 = 2\lambda\epsilon_0\mathcal{E}_0|E_p|^2$ in the linear (slow) growth phase but deviate more in the rapid collapse phase of the wave packet. We note that the collapse of photonic pulses proceeds on a time scale much longer than the photon oscillation period, which is consistent with the slowly varying envelope approximation made in the derivation of Eqs. (2) and (3).

To summarize, we have presented an investigation of the modulational and filamentational instabilities of an intense coherent electromagnetic pulse in a photon gas composed of relativistically hot quasiparticles (photons). For this purpose, we used the recently derived nonlinear system of equations of Marklund et al. [18] and derived a nonlinear dispersion relation in the presence of a constant amplitude photon pump. The dispersion relation reveals that the latter is modulationally unstable against nonresonant finite frequency acoustic perturbations. On the other hand, photon beams are also subjected to a filamentational instability, which can break the photon beams apart and can produce light pipes. We have also performed simulation studies of the Marklund-Brodin-Stenflo equations to understand the long term behavior of the modulationally or filamentationally unstable intense photon beams. In the steady state, we have found a multidimensional cubic Schrödinger equation which admits a cylindrically symmetric photon bullet. Our results exhibit the modulation of the photon wave packet as well as their collapse forming a localized light bullet. The present results should thus help to understand the salient features of nonlinear effects caused by intense short photonic pulses (e.g., gamma ray bursts), while they propagate through the medium which is composed of relativistically hot quasiparticles (photons). Such a situation is common in astrophysical environments [24] as well as in laboratory experiments involving the use of powerful lasers (peak radiation intensity of up to 10^{21} W/cm^2, corresponding to the present peta-Watt laser experiments) which would produce QED phenomena, as described in this Letter.

*Electronic address: bengt@tp4.rub.de
[1] V. I. Karpman, Plasma Phys. **13**, 477 (1971); Phys. Plasmas **5**, 932 (1998).
[2] P. K. Shukla et al., Phys. Rep. **138**, 1 (1986).
[3] R. Bingham, J. T. Mendonça, and J. M. Dawson, Phys. Rev. Lett. **78**, 247 (1997).
[4] J. T. Mendonça, Theory of Photon Acceleration (Institute of Physics, Bristol, 2001).
[5] J. T. Mendonça, J. Phys. A **34**, 9677 (2001).
[6] J. T. Mendonça, Plasma Phys. Controlled Fusion **44**, B225 (2002).
[7] W. Heisenberg and H. Euler, Z. Phys. **98**, 714 (1936).
[8] W. Greiner, B. Müller, and J. Rafelski, Quantum Electrodynamics of Strong Fields (Springer, Berlin, 1985).
[9] V. B. Berestetskii, E. M. Lifshitz, and L. P. Pitaevskii, Quantum Electrodynamics (Pergamon Press, Oxford, 1991), 3rd ed.
[10] D. Mihalas and B. Weibel-Mihalas, Foundation of Radiation Hydrodynamics (Dover Publications, New York, 1999).
[11] Z. Bialynicka-Birula and I. Bialynicki-Birula, Phys. Rev. D **2**, 2341 (1970).
[12] Y. J. Ding and A. E. Kaplan, Phys. Rev. Lett. **63**, 2725 (1989).
[13] N. N. Rozanov, Zh. Eksp. Teor. Fiz. **103**, 1996 (1993) [Sov. Phys. JETP **76**, 991 (1993)].
[14] N. N. Rozanov, Zh. Eksp. Teor. Fiz. **113**, 513 (1998) [Sov. Phys. JETP **86**, 284 (1998)].
[15] G. Brodin, M. Marklund, and L. Stenflo, Phys. Rev. Lett. **87**, 171801 (2001).
[16] G. P. Agrawal, Nonlinear Fiber Optics (Academic, New York, 1989).
[17] G. Brodin, L. Stenflo, D. Anderson, M. Lisak, M. Marklund, and P. Johannison, Phys. Lett. A **306**, 206 (2003).
[18] M. Marklund, G. Brodin, and L. Stenflo, Phys. Rev. Lett. **91**, 163601 (2003).
[19] V. I. Karpman and E. M. Krushkal, JETP **27**, 277 (1969).
[20] V. I. Karpman and H. Washimi, J. Plasma Phys. **18**, 173 (1977); V. I. Karpman et al., Phys. Rev. Lett. **64**, 890 (1990).
[21] P. K. Shukla, Nature (London) **274**, 874 (1978).
[22] Y. Silberberg, Opt. Lett. **15**, 1282 (1990).
[23] M. Desaix, D. Anderson, and M. Lisak, J. Opt. Soc. Am. B **8**, 2082 (1991).
[24] T. Piran, Phys. Rep. **314**, 575 (1999).