Quantum Suppression of the Generic Chaotic Behavior Close to Cosmological Singularities

Martin Bojowald*

*Max-Planck-Institut für Gravitationsphysik, Albert-Einstein-Institut, Am Mühlenberg 1, D-14476 Golm, Germany
and Center for Gravitational Physics and Geometry, The Pennsylvania State University,
104 Davey Lab, University Park, Pennsylvania 16802, USA*

Ghanashyam Date[†]

The Institute of Mathematical Sciences, CIT Campus, Chennai-600 113, India

In classical general relativity, the generic approach to the initial singularity is very complicated as exemplified by the chaos of the Bianchi IX model which displays the generic local evolution close to a singularity. Quantum gravity effects can potentially change the behavior and lead to a simpler initial state. This is verified here in the context of loop quantum gravity, using methods of loop quantum cosmology: The chaotic behavior stops once quantum effects become important. This is consistent with the discrete structure of space predicted by loop quantum gravity.

According to the celebrated singularity theorems of classical general relativity, the backward evolution of an expanding universe leads to a singular state where the classical theory ceases to apply. An extensive analysis of the approach to the singularity, in the general context of inhomogeneous cosmologies, has culminated in the Belinskii-Khalatnikov-Lifschitz scenario [1]. According to this scenario, as the singularity is approached, the spatial geometry can be viewed as a collection of small patches, each of which evolves essentially independently as a homogeneous model, most generally the Bianchi IX model. This is justified by the observation that interactions between the patches are negligible because time derivatives dominate over space derivatives close to a singularity.

The approach to the singularity of a Bianchi IX model is described by a particle moving in a potential with exponential walls (corresponding to the increasing curvature) bounding a triangle (Fig. 1). During its evolution, the particle is reflected at the walls resulting in an infinite number of oscillations (of the scale factors) when the singularity is approached. This classical behavior can be shown to lead to a chaotic evolution by using an analogy with a billiard valid in the asymptotic limit close to the singularity [2].

To appreciate implications of the chaotic approach to the Bianchi IX singularity, observe that at any given time the spatial slice can be decomposed into a collection of almost homogeneous patches, the size of the patches being controlled by the magnitude of space derivatives in the equations of motion. During subsequent evolution when the curvatures grow, these patches have to be subdivided to maintain the homogeneous approximation. This subdivision is also controlled by evolution of the individual patches. Since the patches are homogeneous only to a certain approximation, a subdivision of a given patch at a certain time leads to new patches with slightly

different initial conditions. The chaotic approach to the Bianchi IX singularity then implies that their geometries will depart rapidly from each other, and the patches have to be fragmented more and more the closer one comes to the singularity. This rapid fragmentation suggests a very complicated and presumably fractal structure of the spatial geometry at the classical singularity. Note that both of these features, the breaking up of a spatial slice into approximately homogeneous patches and the unending oscillatory approach to the singularity of individual patches, are ultimately consequences of the unbounded growth of the spatial curvature, i.e., the singularity.

However, the classical evolution towards the singularity is expected to break down when the curvatures become too large and to be replaced by quantum dynamics. If we truncate the classical model at a certain lowest
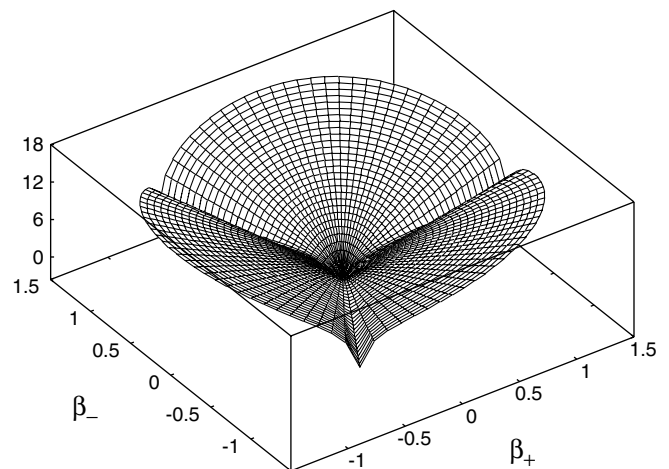

FIG. 1. The potential is shown for $V = 30$ in Planck units. Along the z axis is plotted the logarithm of the potential (shifted so that it is larger than 1 everywhere). Its form does not change with volume due to the factorized volume dependence.

volume, technically chaos does not occur. It is a natural question to ask if the qualitative features of the chaotic approach continue to survive quantum modifications. The answer to this question is neither obvious nor independent of concrete theories of quantum gravity and must be explored within the context of a specific candidate theory.

For instance, effective actions with extra terms and additional fields motivated from string or M theories do not lead to a nonchaotic behavior [3] and, thus, retain their implications for fragmentation in the context of a general inhomogeneous singularity. On the other hand, loop quantum gravity [4,5] predicts a discrete structure of space. For these theories, an unlimited fragmentation of space would be inconsistent with the discrete structure, and one would expect a qualitative modification of the chaotic approach. This translates into a self-consistency test for such theories. *Thus, a quantum theory of gravity with a discrete structure of space must, for self-consistency, provide a mechanism which prevents the unlimited fragmentation in the expected general approach to a classical singularity.*

In this Letter, we use methods of loop quantum cosmology [6–8], a part of loop quantum gravity, to study this issue. This allows us to obtain explicit, nonperturbative modifications of the classical behavior at a small volume which can be analyzed for their implications for chaos. Loop quantum cosmology has already led to a resolution of conceptual problems such as a nonsingular evolution [9] and also given a new scenario for inflation [10] which is based on the small-volume modifications. The proof of absence of singularities extends to homogeneous models, in particular, the Bianchi IX model [7,8]. The investigations of this Letter will provide a new consistency check of loop effects and their physical viability. Specifically, quantum modifications for the vacuum Bianchi IX model (central to the issue of a chaotic approach) are presented and shown to prevent the chaotic behavior.

The dynamics of the Bianchi IX model can be formulated on its minisuperspace spanned by the positive scale factors a_I, $I = 1, \ldots, 3$, related to the diagonal metric components $g_{II} = a_I^2$. It is given by the Hamiltonian constraint

$$H = \frac{2}{\kappa} \left[(\Gamma_J \Gamma_K - \Gamma_I) a_I - \frac{1}{4} a_I \dot{a}_J \dot{a}_K + \text{cyclic} \right], \quad (1)$$

where $\kappa = 8\pi G$ is the gravitational constant and

$$\Gamma_I = \frac{1}{2} \left(\frac{a_J}{a_K} + \frac{a_K}{a_J} - \frac{a_I^2}{a_J a_K} \right) \quad \text{for } \epsilon_{IJK} = 1 \quad (2)$$

are the spin connection components. Thus, the potential term obtained from (1) is given by

$$W(a_1, a_2, a_3) = \frac{1}{2} \left[\sum_I a_I^4 - 2(a_1 a_2 a_3)^2 \sum_I a_I^{-2} \right]. \quad (3)$$

In order to diagonalize the kinetic term of (1), one can introduce the Misner variables [11]: the logarithmic

volume $\Omega := -\frac{1}{3} \log V = -\frac{1}{3} \log(a_1 a_2 a_3)$ and the anisotropies $\beta_{\pm}$, introduced via $a_1 := e^{-\Omega + \beta_+ + \sqrt{3}\beta_-}$, $a_2 := e^{-\Omega + \beta_+ - \sqrt{3}\beta_-}$, $a_3 := e^{-\Omega - 2\beta_+}$. In terms of these variables, the potential (3) is obtained as

$$\mathcal{W}(\Omega, \beta_+, \beta_-) = \frac{1}{2} e^{-4\Omega} \{ e^{-8\beta_+} - 4e^{-2\beta_+} \cosh(2\sqrt{3}\beta_-) + 2e^{4\beta_+} [\cosh(4\sqrt{3}\beta_-) - 1] \}. \quad (4)$$

Evidently, the Ω dependence factorizes, and one obtains an anisotropy potential $\mathcal{V}(\beta_+, \beta_-)$ which exhibits exponential walls for large anisotropies as displayed in Fig. 1. A typical wall can be derived from the potential by setting $\beta_- = 0$ and taking β_+ to be negative, e.g., $W(a_1, a_1, V/a_1^2) = \frac{1}{2} e^{-4\Omega} (e^{-8\beta_+} - 4e^{-2\beta_+}) \sim \frac{1}{2} e^{-4\Omega - 8\beta_+}$.

The wall picture implies that the Universe, described by a particle moving in such a potential, runs through almost free (Kasner) epochs where the potential can be ignored, interrupted by reflections at the walls where the expansion/contraction behavior of different directions changes. The infinite number of these reflections implies that the system behaves chaotically.

One can see that the infinite height of the walls is a consequence of the diverging intrinsic curvature in Γ_I for small and large a_I. In the classical evolution, each a_I will eventually become arbitrarily small as the singularity is approached, e.g., a_1 and a_2 for the typical wall above. However, at a certain stage of the evolution, quantum gravity is expected to become important, which will lead to a modification of the behavior. Clearly, a necessary condition for the chaotic behavior to be prevented by the quantum modification is that quantum gravity should effectively contain an upper limit on the curvature such that the walls would have only finite height, changing the whole scenario.

A quantum theory where such a maximal curvature *follows* is loop quantum cosmology [12,13]. The origin of this upper bound for the curvature lies in the quantization of the relevant quantities and can be compared conceptually to the finite ground state energy of the hydrogen atom obtained after quantization. In the classical equations of motion, one can incorporate this feature of loop quantum cosmology by replacing the Γ_I with effective coefficients which are derived from the quantization [8]. This leads to the corresponding effective potential.

The new, effective potential is more complicated than the original one. Nonetheless, for a large volume, the effective potential approximates the original one. It is significantly different for small volumes and is responsible for the breakdown of chaos. The volume dependence does not factorize, making the analysis of the classical motion (with volume as internal time) harder. One should also keep in mind that the analysis done here uses only the effective classical description which includes some *nonperturbative* quantum effects in the potential. The

full quantum evolution is much more complicated and is given by a partial difference equation for the wave function [8].

Using the effective potential in classical equations of motion is sufficient to shed light on the above consistency requirement, even though it is not valid close to the classical singularity. The underlying dynamical equation of loop quantum cosmology contains a parameter, j, which appears as a quantization ambiguity [14]. Its value controls the size of the Universe where the maximal curvature is attained. By choosing it to be sufficiently large, one can move the quantum effects in the effective potential into the semiclassical domain. Those large values may not be expected or natural from a physical point of view, but they allow us to study the consistency issue in a simplified setting. A necessary requirement for consistency of the quantum theory then is that the effective classical description stops the Bianchi IX oscillations at some volume.

To be more explicit, one first has to introduce densitized triad variables $|p^I| = V/a_I$, with the volume $V = a_1 a_2 a_3 = \sqrt{|p^1 p^2 p^3|}$, which become basic operators in loop quantum gravity. For convenience, we have taken them to be dimensionless by setting $\frac{1}{2}\gamma\ell_P^2 = 1$ in the notation of [8]. Their inverses $|p^I|^{-1}$ do have well-defined quantizations as finite operators, despite the classical curvature divergence at $p^I = 0$ which is eliminated by quantum effects [12]. One can model its effect in the classical behavior by replacing $(p^I)^{-1}$ in the spin connection components by a function $F(p^I/2j)$ to get an effective spin connection which includes nonperturbative quantum effects. The function F is derived directly from a quantization [14]. The ambiguity parameter j appears explicitly and controls the peak of F.

The potential W then gets replaced by an effective one, W_j, by using the effective instead of the classical spin connection components. One can again obtain the typical wall behavior, this time by evaluating $W_j[p^1, p^1, (V/p^1)^2]$ as a function of p^1 at fixed volume. At these arguments, with p^1 large enough but V/p^1 not too small, the complicated expression simplifies to just two dominant terms. As a function of p^1 and the volume, the typical wall becomes (see Fig. 2)

$$W_j(p^1, p^1, 2jq) \approx \frac{V^4 F^2(q)}{32 j^4 q^2}\{3 - 2qF(q)\}, \qquad (5)$$

where we have used the notation $q := \frac{1}{2j}(V/p^1)^2$.

For values at the peak or larger, $F(q) \sim q^{-1}$, and the classical wall $\frac{1}{2}e^{-4\Omega - 8\beta_+}$ is reproduced. The peak of the finite walls is reached for a constant argument q of F which in Misner variables implies that $e^{-2\Omega + 2\beta_+}$ is constant. Thus, the wall maxima lie on the line $\beta_+ = \Omega +$ const in the classical phase space, and the height of the wall drops off as $e^{-12\Omega} \propto V^4$ with decreasing volume. At a very small volume, however, the walls collapse even more rapidly, and a numerical analysis shows that the
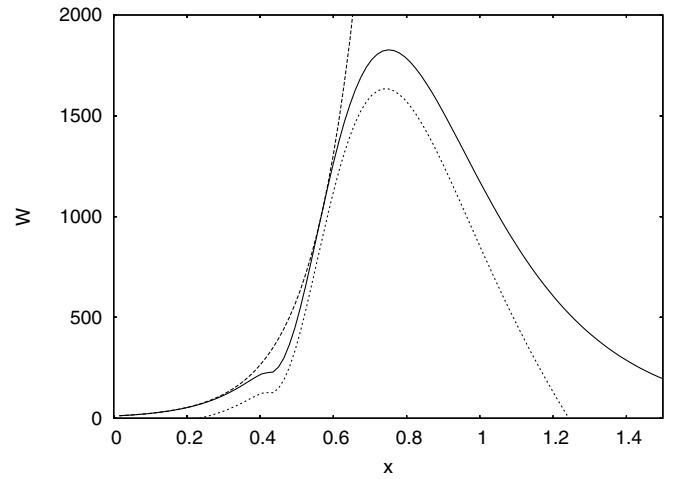
FIG. 2. Effective wall (5) of finite height for $j = 1$, $V = 10$ as a function of $x = -\beta_+$, compared to the classical exponential wall (upper dashed curve). Also shown is the exact wall $W_j[p^1, p^1, (V/p^1)^2]$ (lower dashed curve), which for x smaller than the peak value coincides well with the approximation (5) up to a small, nearly constant shift.

potential becomes negative everywhere at a dimensionless volume of about $(2.172j)^{3/2}$ in Planck units, i.e., just around the elementary discrete volume for the smallest value $j = 1/2$.

In Fig. 3 are shown three snapshots of the effective potential at decreasing volumes in the vicinity of the isotropy point in the anisotropy plane. The potential at larger volumes clearly exhibits a wall (positive potential) of finite height and finite extent. As the volume is decreased, the wall moves *inwards* and its height decreases. Progressively, the wall disappears completely making the potential negative everywhere. Eventually, the potential approaches zero from below.

This immediately shows that the classical reflections will stop after a finite amount of time, rendering the classical argument about chaos inapplicable. The volume where the transition from the classical behavior to the modified behavior takes place, i.e., the first time the Universe can "jump over the wall," depends on the initial conditions but it will certainly happen—the latest at the elementary discrete volume for the smallest value of the ambiguity parameter.

Starting at large volumes, the Universe will pass through several phases of Kasner evolution punctuated by reflections. This will continue until the small-volume modification of the effective potential comes into play. The stability of a Kasner trajectory in the presence of the modified potential will decide the nature of subsequent deviations after the "last" Kasner epoch. Pleasingly, the Kasner trajectories turn out to be stable in the presence of the effective potential [15]. By contrast, these are unstable in the presence of the classical potential.

Intuitively, this gives us a natural and consistent picture of the approach to a classical singularity in a quantum theory with a discrete structure of space: At a larger
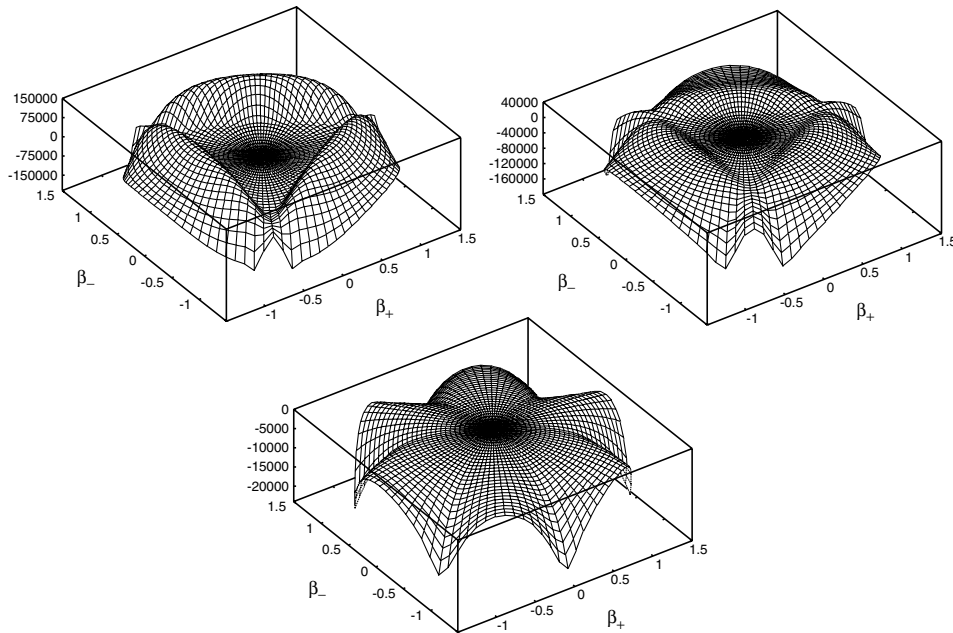
FIG. 3. The modified effective potential is plotted for $j = 5$ at volumes 150, 100, and 30 in Planck units for $-1.5 \leq \beta_{\pm} \leq 1.5$.

volume, the system follows the complicated classical evolution with different Kasner phases interrupted by reflections at the walls. As described previously, in the context of an inhomogeneous space this leads to a fragmentation into smaller and smaller patches. Once quantum effects are taken into account, the reflections stop just when the volume of a given patch is about the size of a Planck volume, i.e., at the scale of the discreteness. Below that scale, a further fragmentation does not take place and the discrete structure is preserved. We again emphasize that the modifications used here become important at the latest when the Planck scale is reached, implying consistency with the expectations from a discrete structure. Below this scale, though, the effective classical description should be superseded by the quantum description. We also emphasize that the existence of a maximal curvature is a *consequence* of quantization and *not* put in, in an *ad hoc* manner. The results thus constitute a consistency test for loop quantum gravity.

This also indicates that the results of [7–9,16] which prove a nonsingular quantum evolution of homogeneous models can be generalized to the full theory, removing also inhomogeneous classical singularities. At the present stage, however, the results for the general case are to be regarded as preliminary and have to be supported by more general techniques directly in inhomogeneous quantum models.

*Electronic address: mabo@aei.mpg.de
†Electronic address: shyam@imsc.res.in

[1] V. A. Belinskii, I. M. Khalatnikov, and E. M. Lifschitz, Adv. Phys. **13**, 639 (1982).

[2] T. Damour, H. Henneaux, and H. Nicolai, Classical Quantum Gravity **20**, R145 (2003).

[3] T. Damour and M. Henneaux, Phys. Rev. Lett. **85**, 920 (2000).

[4] C. Rovelli, Living Rev. Relativ. **1**, 1 (1998).

[5] T. Thiemann, gr-qc/0110034.

[6] M. Bojowald and H. A. Morales-Técotl, in Proceedings of the Fifth Mexican School (DGFM): The Early Universe and Observational Cosmology, Lecture Notes in Physics (Springer-Verlag, Berlin, to be published), gr-qc/0306008.

[7] M. Bojowald, Classical Quantum Gravity **20**, 2595 (2003).

[8] M. Bojowald, G. Date, and K. Vandersloot, Classical Quantum Gravity **21**, 1253 (2004).

[9] M. Bojowald, Phys. Rev. Lett. **86**, 5227 (2001).

[10] M. Bojowald, Phys. Rev. Lett. **89**, 261301 (2002).

[11] C. W. Misner, Phys. Rev. Lett. **22**, 1071 (1969).

[12] M. Bojowald, Phys. Rev. D **64**, 084018 (2001).

[13] T. Thiemann, Classical Quantum Gravity **15**, 1281 (1998).

[14] M. Bojowald, Classical Quantum Gravity **19**, 5113 (2002).

[15] M. Bojowald, G. Date, and G. M. Hossain (to be published).

[16] M. Bojowald, Classical Quantum Gravity **19**, 2717 (2002).