# Casimir Effect and Geometric Optics

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We propose a new approach to the Casimir effect based on classical ray optics. We define and compute the contribution of classical optical paths to the Casimir force between rigid bodies. We reproduce the standard result for parallel plates and agree over a wide range of parameters with a recent numerical treatment of the sphere and plate with Dirichlet boundary conditions. Our approach improves upon the proximity force approximation. It can be generalized easily to other geometries, other boundary conditions, to the computation of Casimir energy densities, and to many other situations.

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Improvements in experimental methods have rekindled efforts to compute Casimir forces for geometries beyond the classic case of parallel plates [1,2]. No exact expressions are known even for simple geometries such as two spheres or a sphere and a plane. It is therefore interesting to consider new ways of viewing the Casimir effect and the approximation schemes that they motivate. In this Letter we present a new approach based on classical ray optics. Our approach avoids the infinities that have plagued Casimir calculations. Like ray optics it is most accurate at short wavelengths and where diffraction is not important. Our basic result, see Eq. (6), is simple and easy to implement. It coincides with the well-known proximity force approximation (PFA) [3] close to the parallel plates limit. Recently a precise numerical result has been obtained for the Dirichlet Casimir energy of a sphere of radius $R$ separated from a plane by a distance $a$ [4]. This provides us an opportunity to test our approximation. The results are shown in Fig. 1. They give us encouragement that the optical approach may provide a useful tool for estimating Casimir forces in situations where exact calculations are not available.

We consider a scalar field of mass $m$ satisfying the wave equation, $(-\nabla^2 - k^2)\phi(x) = 0$, in a domain $\mathcal{D} \subset \mathbb{R}^3$ bounded by disconnected surfaces, $S_1, S_2, \ldots,$ on which it obeys Dirichlet (or Neumann) boundary conditions. At the end we comment on the generalization to conducting boundary conditions for the electromagnetic field. The Casimir energy can be written as an integral over $\delta\rho(k)$, the difference between the density of states in $\mathcal{D}$ and the density of states in vacuum. This, in turn, can be related to an integral over the Green's function [5]

$$\mathcal{E} = \frac{1}{2}\hbar \int_0^\infty dk\omega(k)\frac{2k}{\pi}\mathrm{Im}\int_{\mathcal{D}} d^3x\tilde{G}(x, x, k + i\epsilon), \quad (1)$$

where $\omega(k) = \sqrt{c^2k^2 + m^2c^4/\hbar^2}$ and $\tilde{G} \equiv G - G_0$ is the difference between the Green's function and the vacuum Green's function. We work in three dimensions although the generalization of our results to other dimensions is straightforward.

We look for an approximate solution of the wave equation that becomes exact for infinite, planar surfaces. We approximate the full propagator as a sum of terms ascribable to different optical paths, satisfying the wave equation with an error of $\mathcal{O}(1/(kR)^2)$ [6,7] where $R$ is a typical curvature of the surface (e.g., the radius of the sphere in the sphere + plane situation). We are also neglecting diffractive contributions, arising from the presence of sharp boundaries, and we assume the absence of caustics in the integration domain. We expect this to give an excellent approximation for the Casimir energy when $\kappa R$ is large (here $\kappa$ is the dominant wave number in the Casimir energy integral, $\kappa \sim 1/a$ where $a$ is the minimum distance between the surfaces).

The optical contribution to $G(x', x, k)$ is given by the sum over optical paths from $x$ to $x'$, $G(x', x, k) \to G_{\mathrm{optical}}(x', x, k) = \sum_{\mathbf{n}} G_{\mathbf{n}}(x', x, k)$. These fall into classes $C_n$ which have $n$ points on the boundaries [8]. The collective index $\mathbf{n} = (n, \alpha)$ identifies both the class $C_n$ and the path $\alpha$. These paths are stationary points in the class $C_n$ of the functional integral representation of $G$. In three dimensions the optical terms in $G$ contribute

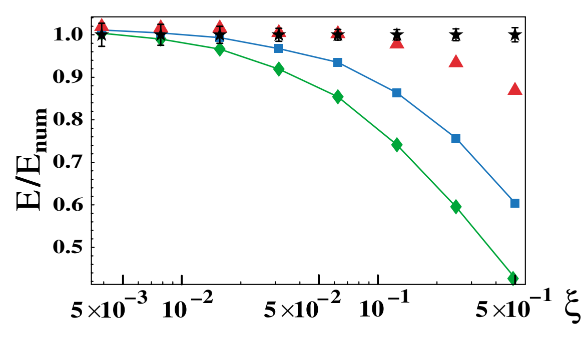


FIG. 1 (color). Comparison of methods: Casimir energy for a sphere and a plate as a function of $\xi = a/R$. Numerical data from Ref. [4] normalized to unity (stars with error bars); our optical approximation (red triangles); plate based proximity force approximation (PFA) (green diamonds); sphere based PFA (blue squares).

  

$$G_{\rm optical}(x', x, k) = \frac{1}{4\pi}\sum_{\mathbf{n}}(-1)^n\sqrt{\Delta_{\mathbf{n}}(x', x)}e^{ik\ell_{\mathbf{n}}(x',x)} \quad (2)$$

(in $d \neq 3$ Hankel functions will appear in the analogous expression [9]), where $\ell_{\mathbf{n}}(x', x)$ is the length of the optical path $\mathbf{n} = (n, \alpha)$ that starts from $x$ and arrives at $x'$ after reflecting $n$ times from the boundary. These paths are the minima of $\ell_{\mathbf{n}}(x', x)$, straight lines that reflect with equal angles of incidence and reflection from the surfaces. The factor $(-1)^n$ implements the Dirichlet boundary condition. For Neumann boundary conditions it is absent. $\Delta_{\mathbf{n}}(x', x)$ is the enlargement factor of classical ray optics [7] (also equal to the VanVleck determinant as defined in Ref. [9]) and is given by

$$\Delta_{\mathbf{n}}(x', x) = \frac{d\Omega_x}{dA_{x'}} = \lim_{\delta\to 0}\delta^{-2}e^{-\int_\delta^\ell ds[(1/R_1)+(1/R_2)]}, \quad (3)$$

where $R_{1,2}(s)$ are the radii of curvature of the wave front following the path and $s$ is the coordinate along the path. $\Delta$ measures the spread in area $dA$ at the arrival point $x'$ of a pencil of rays having angular width $d\Omega$ at the starting point $x$, following the classical path indexed by $\mathbf{n}$. This is reasonably easy to compute even for multiple specular reflections on any curved surface.

The contribution of the optical path $\mathbf{n}$ to the Casimir energy is obtained by substituting Eq. (2) into Eq. (1),

$$\mathcal{E}_{\mathbf{n}} = (-1)^n M_n \int_0^\infty \frac{dk}{4\pi^2}\hbar k\omega(k)\int_{\mathcal{D}_{\mathbf{n}}} d^3x\sqrt{\Delta_{\mathbf{n}}(x)}\sin k\ell_{\mathbf{n}}(x), \quad (4)$$

where $M_n$ is the multiplicity of the $n$th path, and we have defined $\Delta_{\mathbf{n}}(x) = \Delta_{\mathbf{n}}(x, x)$ and $\ell_{\mathbf{n}}(x) = \ell_{\mathbf{n}}(x, x)$ for brevity. The minimum in the class $C_0$ (the direct path for $x$ to $x'$) should be excluded to account for the subtraction of the vacuum energy. In a given geometry the optical paths can be indexed according to the number of reflections from each surface. For example in a geometry consisting of only two convex plates ($S_1$ and $S_2$) we have paths reflecting once on $S_1$ or once on $S_2$, paths reflecting 2 times (once on $S_1$ and once on $S_2$), and so on. The multiplicity of even reflections paths is 2 (the path can be run in two different directions), while that of odd reflections is 1. $\mathcal{D}_{\mathbf{n}}$ is the domain over which the path $\mathbf{n} = (\mathbf{n}, \alpha)$ is possible.

$\mathcal{E}_{\mathbf{n}}$ given by Eq. (4) diverges if paths of arbitrary small length occur. For domains bounded by convex plates $\ell_{\mathbf{n}}(x, x) \to 0$ can occur only for the first reflection, $n = 1$. To regulate this divergence we separate the initial and final points by a distance $\epsilon$, so that $\ell_{\mathbf{n}} \geq \epsilon$. This is equivalent to putting a cutoff on the frequency at $k \sim 1/\epsilon$ [10,11]. Because it is confined in the first reflection, the divergence will never contribute to the force between surfaces. In practice it can be isolated and discarded. Next we interchange the integrals over $k$ and $x$ in Eq. (4), and perform the $k$ integral,

$$\mathcal{E}_{\mathbf{n}} = (-1)^{n+1}M_n\frac{m^2c^3}{4\pi^2\hbar}\int_{\mathcal{D}_{\mathbf{n}}} d^3x\frac{\sqrt{\Delta_{\mathbf{n}}(x)}}{\ell_{\mathbf{n}}(x)}K_2(mc\ell_{\mathbf{n}}(x)/\hbar). \quad (5)$$

For a massless scalar we let $m \to 0$ and we obtain our fundamental result,

$$\mathcal{E}_{\rm optical} = -\frac{\hbar c}{2\pi^2}\sum_{\mathbf{n}}(-1)^n M_n\int_{\mathcal{D}_{\mathbf{n}}} d^3x\frac{\sqrt{\Delta_{\mathbf{n}}(x)}}{\ell_{\mathbf{n}}^3(x)} \quad (6)$$

which expresses the optical approximation to the Casimir effect as a sum over geometric quantities alone.

Our method should not be confused with Gutzwiller's semiclassical approximation to the density of states [12,13], nor with Balian's and Bloch's multiple reflection expansion for Green's function [10]. The latter expresses the Green's function in terms of surface integrals, not limited to the optical paths. The former corresponds to performing the integration over $x$ in Eq. (4) by stationary phase and selects only periodic paths. This approximation fails badly when the radius of curvature $R$ of the surface(s) is large compared both to the separation $a$ and the width $L$ of the surfaces. Our method applies also in situations in which no periodic classical paths exist.

Parallel plates provide a simple, pedagogical example which has many features — fast convergence, trivial isolation of divergences, dominance of the even reflections — that occur in all the geometries we have analyzed. We assume for simplicity that the two plates have the same area $S$. The relevant paths are shown in Fig. 2 where the points $x$ and $x'$, which should coincide, are separated for ease of viewing. For the even paths $\ell_{2n}(z) = 2na$, $n = 1, 2, \ldots$, independent of $z$ (here $z$ is the distance from the lower surface). For the odd paths $\ell_{2n-1,\alpha}(z) = 2(n-1)a + 2\zeta$, where $\zeta = z, a - z$, respectively, if $\alpha =$ down, up, $n = 1, 2, \ldots$. For planar boundaries the enlargement factor is given by $\Delta_{\mathbf{n}} = 1/\ell_{\mathbf{n}}^2$.

The sum over even reflections,

$$\mathcal{E}_{\rm even} = -\frac{\hbar c}{\pi^2}\sum_{n=1}^\infty\int_{S_d} dS\int_0^a dz\frac{1}{(2na)^4} = -\frac{\pi^2\hbar c}{1440a^3}S \quad (7)$$

is trivial because it is independent of $z$. The result is the

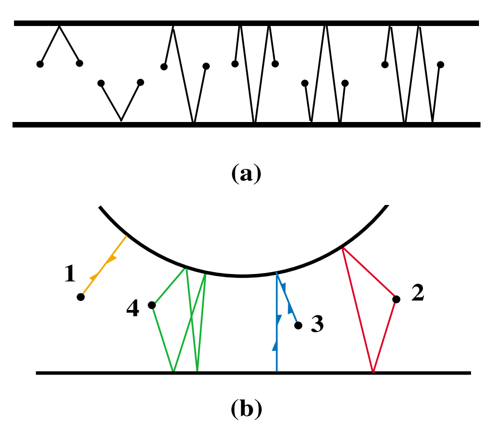


FIG. 2 (color online). (a) Optical paths for parallel plates. Initial and final points have been separated for visibility; (b) optical paths for plane + sphere. For $n = 1$ a reflection off the sphere is shown; for $n = 3$ a reflection twice off the sphere is shown.

usual Dirichlet Casimir energy [2]. The sum over odd reflections gives

$$\mathcal{E}_{\rm odd} = \frac{\hbar c}{2\pi^2} \int dS \sum_{n=0}^{\infty} \int_0^a dz \frac{1}{[\epsilon^2 + (2z + 2na)^2]^2} = \frac{\hbar c}{16\pi^2} \frac{2\pi S}{\epsilon^3}. \quad (8)$$

This divergence can be related to that expected on the basis of Ref. [10] and will be discussed in Ref. [17]. For now it suffices that it is proportional to $S$, independent of $a$, does not contribute to the Casmir force, and can be ignored. The fact that the odd reflections sum to a divergent constant is universal for geometries with planar boundaries and to a good approximation is also valid for curved boundaries. Notice that in this situation our method coincides with the method of images [14]. This will not occur for other examples. In Ref. [13] the Casimir energy is approximated as a sum over periodic classical paths [12]. In generic geometries the optical paths are closed but not periodic and will provide a much better approximation to the Casimir energy. The approach of Ref. [13] yields Eq. (7) and coincides with the optical approach in the case of parallel plates.

Note that the sum over $n$, Eq. (7), converges rapidly: 92% of the effect comes from the first term (the two reflection path) and $>98\%$ comes from the two and four reflection paths. This rapid convergence persists for all the geometries we analyzed due to the rapid increase in the length of the paths. Also notice that for $m > 0$ the two reflection contribution gives a uniform approximation to the $a$ dependent part of the Casimir energy (accurate from 92% for $ma \ll 1$ to exponentially small terms for $ma \gg 1$): $\mathcal{E}|_{\rm finite} \simeq \mathcal{E}_2 = -(m^2c^3/8a^2\pi^2\hbar)K_2(2mca/\hbar)$.

The optical approach sheds light on the proximity force approximation, which has been used for years to estimate Casimir forces for geometries in which an exact calculation is unavailable [3]. For the Dirichlet problem and two bodies $S_1$ and $S_2$ it takes the form:

$$\mathcal{E}_{\rm PFA}(S_1) = -\frac{\pi^2\hbar c}{1440} \int_{S_1} dS \frac{1}{[d_{12}(x)]^3}, \quad (9)$$

where $d_{12}(x)$ is the distance from $S_1$ to $S_2$ along the normal to $S_1$ at $x$. The PFA is ambiguous because a different result is obtained by interchanging surfaces $S_1$ and $S_2$. The PFA can be viewed as the sum over optical paths if at each point $x$ in $\mathcal{D}$ the path is chosen normal to $S_1$, and the small area element that intersects this path on $S_2$ is replaced by a plane normal to the path. Then all paths that bounce back and forth between these two parallel surfaces are summed. Clearly the PFA misses three important effects that are correctly included in the optical approach: (i) the actual optical paths are shorter; (ii) the surfaces are curved; and (iii) there are optical paths through points that do not lie on straight lines normal to one surface or the other. Effects (i) and (iii) increase and (ii) decreases for convex (increases for concave) surfaces the optical estimate of the absolute value of the Casimir energy relative to the PFA. In the cases we have studied the net effect is to increase the Casimir energy. In the subsequent discussions we compare our results with the PFA approximations based on either of the two surfaces. First, however, we propose an "optimal" PFA motivated by the optical approach: at each point in $\mathcal{D}$ choose the unique shortest path from $S_1$ to $S_2$ (of length $\ell_{12}$). Replace both surfaces locally by planes perpendicular to this path and sum all optical contributions. The result,

$$\mathcal{E}_{\rm PFA^*}(S) = -\frac{\pi^2\hbar c}{1440} \int_{\mathcal{D}} d^3x \frac{1}{[\ell_{12}(x)]^4}, \quad (10)$$

resolves the ambiguity in the PFA in favor of the shortest paths. Of course, the sum over the actual optical paths including the enlargement factor is more accurate still.

The only nontrivial geometry we know of for which we can test our approximation is a sphere of radius $R$ placed at a distance $a$ from an infinite plane. This has not been solved analytically, but numerical results have been published recently [4]. Defining $\xi = a/R$, we expect the optical approximation to give an error of $\mathcal{O}(\xi^2)$. Certainly, when $\xi \gtrsim 1$ diffraction dominates, the force is given by Casimir and Polder [15], and our optical approximation will fail. Our aim is to study the accuracy of the optical approximation and the domain of its applicability compared, for example, to the PFA.

We have calculated the Casimir energy for this configuration including paths up to four reflections. Some characteristic paths are shown in Fig. 2. The results are plotted in Fig. 1. The $C_1$ and $C_3$ contributions can be evaluated analytically. The divergent contribution from $C_1$ is independent of $a$ and can be put aside. The $a$-dependent, finite parts of $\mathcal{E}_1$ and $\mathcal{E}_3$ are opposite in sign and their sum is always small ($<2\%$) compared to the $\mathcal{E}_2$. $\mathcal{E}_2$ and $\mathcal{E}_4$ can be computed quickly with Mathematica. The optical result agrees with the numerical result of Ref. [4] within error bars ($\sim 1\%$) out to $\xi \approx 0.1$, where the PFA fails badly. Even for $\xi \approx 1$, on the border of its range of validity, $\delta E/E_{\rm N} = 25\%$ where $E_{\rm N}$ is the numerical value given by Ref. [4] and $\delta E = E_{\rm N} - E_{\rm opt}$. In comparison, the "sphere based" PFA gives $\delta E/E_{\rm N} = 58\%$ and the "plate based" PFA $\delta E/E_{\rm N} = 73\%$. (In response to this paper, the authors of Ref. [13] compared their semiclassical approximation with the numerical results of Ref. [4] and found agreement comparable to ours [16]. This can be understood as a consequence of the fact that the effective width of the sphere, $L$, scales like its radius $R$, and is not a general feature of the semiclassical approach.)

The limiting case $\xi \to 0$ is that of two infinite parallel plates. Here the optical approximation must agree analytically (and numerically) with PFA. The agreement is clear in Fig. 1.

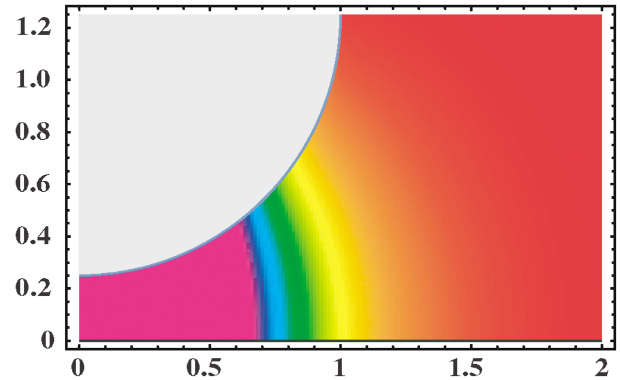


FIG. 3 (color). Local contributions to the optical Casimir energy for a plane and a sphere with $a/R = 0.25$. The scale is linear in the hue from red (least) to violet (greatest). The self-energy density given by $C_1$ has been subtracted.

Since the optical approximation gives the Casimir energy as a volume integral of a local contribution at each point $x$, it is possible to get an idea of the domains that give the dominant contributions to the Casimir force by plotting the integrand in Eq. (6). As an example we show a contour map of the integrand for the plate and sphere at $\xi = 0.25$ in Fig. 3. The local Casimir energy density (and other local observables) is defined by a differential operator acting on the Greens function [11] and is not identical to the integrand in Eq. (6). However, since the form of the optical approximation to the Greens function is so simple, it is straightforward to obtain a compact and computable approximation to the local energy density which would replace Fig. 3.

Our work suggests many possible extensions. We have studied the application to a finite (rectangular) plane inclined at an angle $\theta$ to an infinite plane. This is the geometry of a "Casimir torsion pendulum" [17].

The physically interesting case of conducting boundary conditions can be realized by constructing a matrix Green function for the vector potential. For parallel plates the contribution of odd reflections integrates to 0 (for any $a$), reflecting the well-known absence of a $1/\epsilon^3$ divergence for conducting boundary conditions [11]. The even reflections sum to obtain the expected result: $\mathcal{E}_{\rm cond} = 2\mathcal{E}_{\rm Dirichlet}$. The extension to more complicated geometries will be presented in Ref. [17].

There are many interesting cases (both from a theoretical and an experimental point of view) in which diffraction effects become important, especially in situations in which the objects are small compared to their separation (transition between Casimir and van der Waals forces). Diffraction effects are certainly well beyond the PFA approach but can in principle be included in our framework by using Keller's [8] recipe for constructing the diffracted rays contribution to the propagator and then integrating over $k$ to leave an $x$ integral.

The Casimir integral over modes can be generalized to give the partition function for a fluctuating field at finite temperature. The computation is straightforward and gives the thermal properties of the Casimir force, assuming that it remains reasonable to idealize the material by boundary conditions throughout the range of temperatures of interest [18].

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