Complete Spectrum of Kinetic Eigenmodes for Plasma Oscillations in a Weakly Collisional Plasma

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Kinetic eigenmodes of plasma oscillations in a weakly collisional plasma, described by a collision operator of the Fokker-Planck type, are obtained in closed form for initial-value as well as for boundary-value problems. These eigenmodes, which are smooth and compose a complete discrete spectrum, play the same role for weakly collisional plasmas as the Case–Van Kampen modes do for collisionless plasmas.

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Landau damping of plasma oscillations in a collisionless plasma is one of the most fundamental and widely used concepts in plasma physics [1]. It is well known that Landau-damped solutions are not true eigenmodes. The true eigenmodes for a collisionless plasma were obtained by Van Kampen [2] and Case [3]. The Case–Van Kampen eigenmodes exist for every given real frequency (ω) and wave number (k) and, thus, constitute a continuous spectrum. Although the Case–Van Kampen eigenmodes are singular in functional form, they have been shown to obey classical orthogonality and completeness theorems [2,3]. In most situations of physical interest where the initial conditions are smooth, a broad and continuous spectrum of Case–Van Kampen modes is excited. The Landau-damped solutions emerge, in the long-time limit, as remnants due to phase mixing of the spectrum of singular eigenmodes.

How is this widely accepted physical picture of Landau damping modified if collisions are introduced? Lenard and Bernstein (LB) [4] considered the problem using an operator of the Fokker-Planck type [5]. They obtained an exact analytic solution with a dispersion relation that has a root that formally reduces to a Landau root in the limit of zero collisions. However, they did not discuss the nature of the spectrum or address the issue of completeness of the eigenmodes.

The main objectives of this Letter are to give a theoretical formulation that yields these eigenmodes for both initial-value and boundary-value problems and to demonstrate rigorously that they form a complete set. Unlike the Case–Van Kampen eigenmodes, the complete kinetic eigenmodes of the collisional problem are smooth and constitute a discrete spectrum [6,7]. The Landau-damped solutions emerge as true eigenmodes of the weakly collisional theory in the limit of zero collision [7].

Our demonstration of the completeness of the kinetic eigenmodes of a collisional plasma raises possible questions about some classical treatments, now described in textbooks [8]. Su and Oberman (SO) [9] (also [10] and, recently, [11]) claimed that plasma wave echoes [12–14] (and the ballistic response), which owe their existence to the intrinsic time reversibility of the Vlasov equation, should decay very rapidly due to the presence of collisions. Specifically, using the LB collision operator, they predicted that spatial echoes should decay as $\exp[-(\nu/\omega_p\lambda_D^3)x^3]$ and temporal echoes as $\exp[-\nu\omega_p^2t^3]$, where ν is the collision frequency, ω_p is the plasma frequency, and λ_D is the Debye length. Thus, SO raised the concern that "...the collisional damping of the free-streaming motion can be quite important in certain circumstances so as to make it impossible for the generation of plasma echoes to occur" [9]. Our proof of the completeness of collisional eigenmodes implies that the SO solution, if it exists (or is realized as a transient response), must be a result of the superposition of these eigenmodes. The predictions of our theory are qualitatively consistent with recent experiments involving a weakly collisional stable plasma in which the measured decay rate for the least damped electrostatic ion perturbations was found to be substantially weaker and scaled quite differently than predicted by the SO theory [6].

We build on the foundation of our previous work [7]. We assume that the ion distribution is unperturbed and focus on electrostatic electron perturbations. We begin with the one-dimensional linearized equations for the first order electron distribution function $f(x, t, v)$, coupled with the self-consistent Coulomb's law for the electric field:

$$\frac{\partial f}{\partial t} + v\frac{\partial f}{\partial x} - \frac{e}{m}\frac{\partial f_0}{\partial v}E = \nu\frac{\partial}{\partial v}\left(vf + v_0^2\frac{\partial f}{\partial v}\right), \quad (1)$$

$$\frac{\partial E}{\partial x} = -4\pi e\int_{-\infty}^{\infty} dv f(x, t, v). \quad (2)$$

The collision frequency ν is a constant, and $-e$, m, and v_0 are, respectively, the charge, mass, and thermal speed of the electron. The function f_0 is the equilibrium Maxwellian distribution. The right-hand side of Eq. (1) is the linearized collision operator used in [4].

For eigenmode solutions, we consider the form $f(x, t, v) = \tilde{f}(k, \omega, v) \exp[i(kx - \omega t)]$. We give a unified treatment for two types of problems: (a) the temporal evolution problem, for which k is real and ω is complex, and (b) the spatial evolution problem, for which ω is real and k is complex. We define a slightly different set of dimensionless variables in cases (a) and (b). In the following, all equation numbers with a subscript (a) are for case (a) and a subscript (b) are for case (b). Combining Eqs. (1) and (2), we obtain

$$(u - \Omega)g - \eta \int_{-\infty}^{\infty} g du' = -i\mu \frac{\partial}{\partial u}\left(ug + \frac{1}{2}\frac{\partial g}{\partial u}\right), \quad \text{(3a)}$$

for case (a), where $u \equiv v/(\sqrt{2}v_0)$, $\Omega \equiv \omega/(\sqrt{2}kv_0)$, $g(u) \equiv \sqrt{2}v_0 f/n_0$, $g_0 \equiv \exp[-u^2]/\pi^{1/2}$, n_0 is the equilibrium electron density, $\eta(u) \equiv \alpha(\partial g_0/\partial u)/2$, $\alpha \equiv \omega_p^2/(k^2v_0^2) = 4\pi n_0 e^2/(mk^2v_0^2)$, and $\mu \equiv \nu/(\sqrt{2}kv_0)$. For case (b), we obtain

$$\left(u - \frac{1}{\kappa}\right)g - \frac{\eta}{\kappa^2}\int_{-\infty}^{\infty} g du' = -i\frac{\mu}{\kappa}\frac{\partial}{\partial u}\left(ug + \frac{1}{2}\frac{\partial g}{\partial u}\right), \quad \text{(3b)}$$

where $\kappa \equiv \sqrt{2}kv_0/\omega$, and we redefine $\alpha \equiv 2\omega_p^2/\omega^2$, $\mu \equiv \nu/\omega$, keeping all other definitions the same as for case (i). In [7], eigenfunctions and eigenvalues in Eqs. (3a) and (3b) are found numerically, using an expansion in Hermite polynomials, with coefficients found by solving a novel recurrence relation. It is found that the eigenvalues are discrete, unlike the Case–Van Kampen continuous spectrum [2,3]. As the collision frequency tends to zero (that is, $\mu \to 0$), a subset of these eigenvalues tend to the Landau roots determined by the well-known dispersion relation $1 + \alpha[1 + \Omega Z(\Omega)] = 0$, or $1 + \alpha[1 + Z(1/\kappa)/\kappa]/\kappa^2 = 0$, where Z is the plasma dispersion function. However, there are additional roots which tend to values given by $\Omega = -i[n\mu + 1/(2\mu)]$ or $\kappa^2 = 2(i\mu - n\mu^2)$, where n is a non-negative integer, which correspond to the locations of the poles of the dispersion function. Hereafter, we refer to these poles as the LB poles.

A closed form of the eigenfunctions and the dispersion relations can be obtained by solving Eqs. (3a) and (3b) in the Fourier u space [4,7,11], with solutions

$$\tilde{g}_n(w) = -\alpha\left[1 + \frac{i\Omega_n}{\mu} d\left(\frac{1}{2\mu^2} - \frac{i\Omega_n}{\mu}, \frac{1 + \mu w}{2\mu^2}\right)\right]e^{-w^2/4}, \quad \text{(4a)}$$

$$\tilde{g}_n(w) = -\frac{\alpha}{\kappa_n^2}\left[1 + \frac{i}{\mu} d\left(\frac{\kappa_n^2}{2\mu^2} - \frac{i}{\mu}, \frac{\kappa_n^2 + \mu w \kappa_n}{2\mu^2}\right)\right]e^{-w^2/4}, \quad \text{(4b)}$$

where $\tilde{g}_n(w) = \int_{-\infty}^{\infty} g_n(u)e^{iwu}du/\sqrt{2\pi}$ and we choose $\tilde{g}_n(0) = 1$. Define $D(w, \Omega_n) \equiv 1 - \tilde{g}_n(w)$ or $D(w, \kappa_n) \equiv 1 - \tilde{g}_n(w)$. Then the eigenvalues satisfy the dispersion relations $D(\Omega_n) \equiv D(0, \Omega_n) = 0$ or $D(\kappa_n) \equiv D(0, \kappa_n) = 0$. In these equations, $d(a, x) \equiv x^{-a}e^x\gamma(a, x)$ is a single-valued analytic function in the complex a and x planes, except at the simple LB poles when a is a nonpositive integer. [Here $\gamma(a, x) \equiv \int_0^x e^{-t}t^{a-1}dt$ is the incomplete gamma function.] From Eq. (3a), we see that if Ω_n is an eigenvalue, so is $-\Omega_n^*$, with the corresponding eigenfunction $g_n^*(-u)$. Note also that $\mathrm{Im}(\Omega_n) < 0$ for all n, that is, all the roots are damped. Similarly, if κ_n is an eigenvalue, so is $-\kappa_n$, with corresponding eigenfunction $g_n(-u)$. Note that $\kappa_{nr}\kappa_{ni} > 0$ for all n, where $\kappa_{nr} = \mathrm{Re}(\kappa_n)$, $\kappa_{ni} = \mathrm{Im}(\kappa_n)$.

In order to investigate the completeness of these eigenmodes, it is necessary to obtain the solutions to the adjoint equations for Eqs. (3a) and (3b). These adjoint equations are

$$\frac{1}{2}\frac{\partial^2 G_n}{\partial u^2} - u\frac{\partial G_n}{\partial u} - \frac{i}{\mu}(u - \Omega_n)G_n + \frac{i}{\mu}I_n = 0, \quad \text{(5a)}$$

$$\frac{1}{2}\frac{\partial^2 G_n}{\partial u^2} - u\frac{\partial G_n}{\partial u} - \frac{i}{\mu}(u\kappa_n - 1)G_n + \frac{iu}{\mu}I_n = 0, \quad \text{(5b)}$$

where $I_n \equiv \int_{-\infty}^{\infty} \eta(u)G_n(u)du$. By direct substitution, using the relation $\int_{-\infty}^{\infty} g_n(u)du = \Omega_n^{-1}\int_{-\infty}^{\infty} ug_n(u)du$ or $\int_{-\infty}^{\infty} g_n(u)du = \kappa_n\int_{-\infty}^{\infty} ug_n(u)du$, it can be shown easily that $G_n(u) \propto g_n(u)\exp(u^2) + \alpha\int_{-\infty}^{\infty} g_n(u')du'/\pi^{1/2}$ for the temporal problem and $G_n(u) \propto g_n(u)\exp(u^2)$ for the spatial problem. This shows directly that the eigenvalues of Eqs. (5a) and (5b) are identical to those of Eqs. (3a) and (3b). However, to obtain closed-form expressions for the eigenfunctions, we still need the solutions in Fourier u space,

$$\tilde{G}_n(w) = i\sqrt{2\pi}\,\alpha I_n(1 - \mu w)^{\frac{1}{2\mu^2} - \frac{i\Omega_n}{\mu} - 1} e^{\frac{w^2}{4} + \frac{w}{2\mu}}\left[\Theta\left(-w + \frac{1}{\mu}\right) - \Theta(-w)\right], \quad \text{(6a)}$$

$$\tilde{G}_n(w) = \frac{i\sqrt{2\pi}\,\alpha I_n}{\kappa_n^2}\left(1 - \frac{\mu w}{\kappa_n}\right)^{\frac{\kappa_n^2}{2\mu^2} - \frac{i}{\mu} - 1} e^{\frac{w^2}{4} + \frac{w\kappa_n}{2\mu}}\left[\Theta\left(-w + \frac{\kappa_n}{\mu}\right) - \Theta(-w) - i\kappa_n\delta(w)\right], \quad \text{(6b)}$$

where $\Theta(x)$ is the Heaviside step function such that $\Theta'(x) = \delta(x)$. [The step function $\Theta(x)$ can be analytically continued for complex arguments.] Solutions given by Eqs. (6a) and (6b) exist only if Ω_n, κ_n satisfy the dispersion relations. Noting that the eigenvalues for the adjoint equations are the same as those of the original equations, it is easy to prove the following orthogonality relations:

$$(\Omega_m - \Omega_n)\int_{-\infty}^{\infty} g_m(u)G_n(u)du = 0, \tag{7a}$$

$$(\kappa_m - \kappa_n)\int_{-\infty}^{\infty} ug_m(u)G_n(u)du = 0. \tag{7b}$$

We can now fix the coefficients of the eigenfunctions in Eqs. (6a) and (6b) by the usual orthonormality requirement; that is, we require that the integrals in Eqs. (7a) and (7b) become unity when $m = n$. Now, if completeness is assumed, it follows that for an initial-value problem with an arbitrary initial condition $g(u)$ (subjected to suitable smoothness and boundedness conditions), the solution to Eqs. (1) and (2) [or equivalently, Eq. (3a) with $\Omega \to i\partial/\partial t$] is given by $g(u,t) = \sum_n c_n g_n(u)\exp(-i\Omega_n t)\Theta(t)$, where we have omitted the $\exp(ix)$ factor. Similarly, the solution for the boundary-value problem with a sinusoidal driving electric field of the form $E_{\rm ext} = \varepsilon_e \delta(x)$, such that $\partial[E(x) - E_{\rm ext}]/\partial x = -\int_{-\infty}^{\infty} g du$, is given by $g(u,x) = \sum_n c_n g_n(u)\exp(i\kappa_n x)\mathrm{sgn}(\kappa_{ni})\Theta[\mathrm{sgn}(\kappa_{ni})x]$, where we have omitted the $\exp(-it)$ factor. The coefficients c_n are given, respectively, by

$$c_n = \int_{-\infty}^{\infty} g(u)G_n(u)du, \tag{8a}$$

$$c_n = -\frac{\alpha\varepsilon_e}{\sqrt{\pi}}\int_{-\infty}^{\infty} ue^{-u^2}G_n(u)du. \tag{8b}$$

An independent way to obtain these solutions is by solving the initial-value and boundary-value problems directly using Laplace transforms in time or Fourier transforms in space. We define $\tilde{g}(w,t) = \int_L d\Omega e^{-i\Omega t} \times \tilde{g}_L(w,\Omega)/2\pi$, where L denotes the inverse Laplace transform path, which for our case can be taken simply as the real Ω axis, and $\tilde{g}(w,x) = \int_{-\infty}^{\infty} d\kappa e^{i\kappa x}\tilde{g}(w,\kappa)/2\pi$. Note that $\tilde{g}(0,t)$, or $\tilde{g}(0,x)$ is just the electric field. The integrands in the inverse transforms can be shown to be

$$\tilde{g}_L(w,\Omega) = N(\Omega)[1 - D(w,\Omega)]/D(\Omega) + N(w,\Omega), \tag{9a}$$

$$\tilde{g}(w,\kappa) = N(\kappa)[1 - D(w,\kappa)]/D(\kappa) + N(w,\kappa), \tag{9b}$$

with

$$N(w,\Omega) = \int_{-\frac{1}{\mu}}^{w} \frac{\tilde{g}(w')}{1+\mu w'}\left(\frac{1+\mu w}{1+\mu w'}\right)^{\frac{i\Omega}{\mu}-\frac{1}{2\mu^2}} e^{\frac{w'^2-w^2}{4}+\frac{w-w'}{2\mu}}dw', \tag{10a}$$

$$N(w,\kappa) = \frac{i\alpha\varepsilon_e}{\mu\kappa\sqrt{2\pi}}[\mu + id(\frac{\kappa^2}{2\mu^2} - \frac{i}{\mu}, \frac{\kappa^2}{2\mu^2} + \frac{\kappa w}{2\mu})]e^{-\frac{w^2}{4}}, \tag{10b}$$

with $N(\Omega) \equiv N(0,\Omega)$, $N(\kappa) \equiv N(0,\kappa)$. Note that we have been able to find a closed-form expression for $N(w,\kappa)$ because in the boundary-value problem we have already specified the driving electric field to be a δ function. In order to make further analytical progress for the initial-value problem, we need to impose more specific restrictions on $\tilde{g}(w)$. We assume that it is bounded in w space, it decays no slower than $\exp(-w^2/4)$ for large w, and it is a smooth function so that it can be represented by a Taylor series,

$$\tilde{g}(w)e^{\frac{w^2}{4}} \equiv \tilde{h}(w) = \sum_{n=0}^{\infty}\frac{1}{n!}\tilde{h}^{(n)}(-\mu^{-1})\left(w + \frac{1}{\mu}\right)^n. \tag{11}$$

These assumptions imply that the initial condition $g(u)$ is bounded in u space by the asymptotic behavior of $\exp(-u^2)$. Even with the stated restrictions, Eq. (11) can represent a very large class of physically interesting initial conditions. Then Eq. (10a) becomes

$$N(w,\Omega) = \sum_{n=0}^{\infty}\frac{1}{\mu n!}\tilde{h}^{(n)}(-\mu^{-1})\left(w + \frac{1}{\mu}\right)^n \times d\left(n + \frac{1}{2\mu^2} - \frac{i\Omega}{\mu}, \frac{1+\mu w}{2\mu^2}\right)e^{-\frac{w^2}{4}}. \tag{12a}$$

By the properties of the d function, we see that $N(w,\Omega)$ has simple poles that coincide with the LB poles. However, there are no such poles in $\tilde{g}_L(w,\Omega)$ given by Eq. (9a), since the poles in the two terms cancel each other exactly. [This can be shown using the expressions in Eqs. (4a) and (12a).] The same is true for $\tilde{g}(w,\kappa)$ since the two terms in Eq. (9b) can be combined to give

$$\tilde{g}(w,\kappa) = \frac{i\alpha\varepsilon_e}{\mu\kappa D(\kappa)\sqrt{2\pi}}\left[\mu + id\left(\frac{\kappa^2}{2\mu^2} - \frac{i}{\mu}, \frac{\kappa^2}{2\mu^2} + \frac{\kappa w}{2\mu}\right)\right] \times e^{-\frac{w^2}{4}}. \tag{12b}$$

In Eq. (12b), the LB poles in the numerator cancel the poles in the denominator $D(\kappa)$ since the locations of the LB poles do not depend on w. Therefore, the only poles in the integrands of the inverse transforms are the zeros of $D(\Omega)$ and $D(\kappa)$.

We now show that we can evaluate the inverse transforms using the residue theorem by closing the integration contour in the lower Ω plane or in the upper (lower) κ plane for positive (negative) x. To do so, we need to determine the asymptotic behavior of the integrands for large Ω, or large κ. For the initial-value problem, we can make use of the identity [16]

$$d(a,x) = e^x\sum_{n=0}^{\infty}\frac{(-x)^n}{(a+n)n!} \overset{|a|\to\infty}{\longrightarrow} \frac{1}{a} + \frac{x}{a^2} + \cdots, \tag{13a}$$

where the limit $|a| \to \infty$ is valid for finite x if $a \neq$ negative integer (corresponding to LB poles in our case). Using this identity, we obtain $1 - D(w,\Omega) \to i\alpha w\exp(-w^2/4)/2\Omega$, $D(\Omega) \to 1$, $N(w,\Omega) \to i\tilde{h}(w) \times \exp(-w^2/4)/\Omega$, $N(\Omega) \to i\tilde{h}(0)/\Omega$, so that $\tilde{g}_L(w,\Omega) \to i\tilde{h}(w)\exp(-w^2/4)/\Omega$, as $|\Omega| \to \infty$. Therefore, because of the $\exp(-i\Omega t)$ factor for $t > 0$, the integration contour can, indeed, be closed in the lower Ω plane. For the boundary-value problem, Eq. (13a) cannot be used directly since both a and x are large for large κ. Fortunately, there are more specific asymptotic expansions for the incomplete gamma function in the mathematical literature [17–19]. One of the relations which is relevant to our problem is [17]

$$\frac{\gamma[a+1, a+y(2a)^{1/2}]}{\Gamma(a+1)} = \frac{\mathrm{erfc}(-y)}{2} - \sqrt{\frac{2}{a\pi}}\frac{1+y^2}{3}e^{-y^2} + O(a^{-1}), \qquad (13b)$$

where erfc denotes the complementary error function [16], and $a \sim \kappa^2/2\mu^2$, $y \sim w/2$ as $a \to \infty$. Note that by Stirling's formula for the gamma function, $\Gamma(a) \to e^{-a}a^{a-1/2}\sqrt{2\pi}(1 + 1/12a + \ldots)$, we obtain $x^{-a}e^x\Gamma(a) \to \exp[(x-a)^2/2a]\sqrt{2\pi/a}$, as $|a| \sim |x| \to \infty$. Equation (13b) was originally derived for finite real y and a large positive real a. More general expressions, valid for the whole complex a plane, are given in [18,19], and have the same asymptotic behavior as given by Eq. (13b). From these relations, we conclude that the d functions in Eqs. (4b), (10b), and (12b) all decay as $O(\kappa^{-1})$. Equation (13b) is also useful in obtaining the limiting form of the dispersion relations for the case of a finite Ω or κ as $\mu \to 0$. It is easy to verify that $D(\Omega) \to 1 + \alpha[1 + \Omega Z(\Omega)]$ and $D(\kappa) \to 1 + \alpha[1 + Z(1/\kappa)/\kappa]/\kappa^2$, the Landau-damped solutions, which should be distinguished from the solutions that correspond to the LB poles. Therefore, using these asymptotic relations, we see that $D(\kappa) \to 1$, $\tilde{g}(w, \kappa) \to i\alpha\varepsilon_e(2\pi)^{-1/2}\exp(-w^2/4)/\kappa$, as $|\kappa| \to \infty$. Again, because of the $\exp(i\kappa x)$ factor, the integration contour can, indeed, be closed in the upper (lower) κ plane for positive (negative) values of x. Since the LB poles cancel out, the only residues are the zeros of $D(\Omega)$ and $D(\kappa)$. Thus, the inverse transforms can be integrated out, summing up the contribution from these residues, to give

$$\tilde{g}(w, t) = -i\sum_n \frac{N(\Omega_n)\tilde{g}_n(w)}{\partial D(\Omega_n)/\partial\Omega} e^{-i\Omega_n t}\Theta(t), \qquad (14a)$$

$$\tilde{g}(w, x) = i\sum_n \frac{N(\kappa_n)\tilde{g}_n(w)}{\partial D(\kappa_n)/\partial\kappa} e^{i\kappa_n x}\mathrm{sgn}(\kappa_{ni})\Theta[\mathrm{sgn}(\kappa_{ni})x], \qquad (14b)$$

where we have used Eqs. (4a) and (4b). So, the solutions can, indeed, be expressed as a linear superposition of the eigenfunctions. Using the orthonormality relations, it can be shown by straightforward but tedious algebra that the coefficients in Eqs. (14a) and (14b) are given by Eqs. (8a) and (8b). Therefore, the result from the direct calculation is just the same as the result obtained by assuming completeness of the set of eigenfunctions. We have thus shown that the eigenfunctions for both the initial-value and boundary-value problems form a complete set.

So far our considerations have been based on a smooth initial condition, given by Eq. (11). It is now interesting to consider a singular initial condition, such as $\tilde{g}(w) = \delta(w - w_1)$, with $w_1 > 0$. Then by Eq. (10a), $N(\Omega) = 0$. The solution in u space is given by

$$g(u, t) = \exp\left[\frac{t(2\mu^2 - 1)}{2\mu} - iuw_1' + \frac{w_1^2 - w_1'^2}{4} + \frac{w_1' - w_1}{2\mu}\right]\Theta(t)/\sqrt{2\pi},$$

where $w_1' = [(1 + \mu w_1)\exp(\mu t) - 1]/\mu$. This is an exact solution to the initial-value problem of Eqs. (1) and (2). However, it is an unphysical solution since it does not fall off as $\exp(-u^2)$ for large $|u|$. In the limit of $\mu \to 0$, $g(u, t) \to \exp[-iut - \mu t^3/6 + \ldots]\Theta(t)/\sqrt{2\pi}$. This has the same decaying factor as the SO solution [9], that is, $g(u, t) \propto \exp[-u^2 - iut - \mu t^3/6]$. The conditions under which a general solution reduces to SO form, as it must, remains a subject for future investigation.

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[1] L. Landau, J. Phys. (Moscow) **10**, 25 (1946).
[2] N. G. Van Kampen, Physica (Amsterdam) **21**, 949 (1955).
[3] K. M. Case, Ann. Phys. (N.Y.) **7**, 349 (1959).
[4] A. Lenard and I. B. Bernstein, Phys. Rev. **112**, 1456 (1958).
[5] S. Chandrasekhar, Rev. Mod. Phys. **15**, 1 (1943).
[6] F. Skiff, S. De Souza-Machado, W. A. Noonan, A. Case, and T. N. Good, Phys. Rev. Lett. **81**, 5820 (1998).
[7] C. S. Ng, A. Bhattacharjee, and F. Skiff, Phys. Rev. Lett. **83**, 1974 (1999).
[8] T. H. Stix, *Waves in Plasmas* (American Institute of Physics, New York, 1992).
[9] C. H. Su and C. Oberman, Phys. Rev. Lett. **20**, 427 (1968).
[10] V. I. Karpman, Sov. Phys. JETP **24**, 603 (1967).
[11] R.W. Short and A. Simon, Phys. Plasmas **9**, 3245 (2002).
[12] R.W. Gould, T. M. O'Neil, and J. H. Malmberg, Phys. Rev. Lett. **19**, 219 (1967).
[13] T. M. O'Neil, Phys. Fluids **11**, 2420 (1968).
[14] T. H. Jensen, J. H. Malmberg, and T. M. O'Neil, Phys. Fluids **12**, 1728 (1969).
[15] M. N. Rosenbluth, W. M. MacDonald, and D. L. Judd, Phys. Rev. **107**, 1 (1957).
[16] M. Abramowitz and I. Stegun, *Handbook of Mathematical Functions* (National Bureau of Standards, Washington, DC, 1964).
[17] F. G. Tricomi, Math. Z. **53**, 136 (1950).
[18] N. M. Temme, Math. Comput. **29**, 1109 (1975).
[19] R. B. Paris, J. Comput. Appl. Math. **148**, 323 (2002).