Galactic Potentials

Kayll Lake*
Department of Physics, Queen's University, Kingston, Ontario, Canada K7L 3N6
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The information contained in galactic rotation curves is examined under a minimal set of assumptions. If emission occurs from stable circular geodesic orbits of a static spherically symmetric field, with information propagated to us along null geodesics, observed rotation curves determine galactic potentials without specific reference to any metric theory of gravity. Given the potential, the gravitational mass can be obtained by way of an anisotropy function of this field. The gravitational mass and anisotropy function can be solved simultaneously in a Newtonian limit without specifying any specific source. This procedure, based on a minimal set of assumptions, puts very strong constraints on any model of the "dark matter."

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There now seems to be a widespread belief that "dark matter" is a major constituent of our Universe [1]. Whereas the need for dark matter in some galactic halos has a long history [2], it is fair to say that at present we are quite far away from a universally accepted solution to the dark matter problem. Indeed, empirically motivated modifications of Newtonian dynamics have been developed as an alternative to dark matter [3]. Despite the profound implications of the problem, it is essentially trivial to understand. Assuming the nonrelativistic Doppler effect and emission from stable circular orbits in a Newtonian gravitational field, it follows that $V^2 \propto [M(r)]/r$, where V is the orbital speed and $M(r)$ is the dynamical mass. Since V^2 levels off at large r for many galactic halos, M must continue to grow similar to r. Since in many cases the observed galactic components do not produce this growth, what unseen material does?

The approach used here breaks the problem down into three steps: (i) the determination of the galactic potential; (ii) the construction of the effective gravitational mass from this potential with the aide of an anisotropy function; (iii) the simultaneous solution of the effective gravitational mass and the anisotropy function. Here we completely solve steps (i) and (ii) for all metric-type theories of gravity under a minimal set of assumptions. Assuming only that emission occurs from stable timelike circular geodesic orbits in a static spherically symmetric metric with information propagated to us along null geodesics, it is shown that the potential follows directly from observed galactic rotation curves without any specific reference to a theory of gravity. Further, without specifying any model of the background, the introduction of an anisotropy function allows the determination of the effective gravitational mass without using Einstein's equations. Step (iii) is completed with the aide of a Newtonian limit and the nonrelativistic Doppler effect. This last step makes it clear that the dynamical mass M is not the effective gravitational mass against which observed galactic components should be compared to see if indeed any mass is "missing" [4].

First, we construct stable circular timelike geodesic orbits in a static spherically symmetric field. In terms of "curvature" coordinates, the field takes the form [5]

$$ds^2 = \frac{dr^2}{1 - \frac{2m(r)}{r}} + r^2 d\Omega^2 - e^{2\Phi(r)} dt^2, \tag{1}$$

where $d\Omega^2$ is the metric of a unit sphere. The use of curvature coordinates plays no essential role in what follows as it provides merely a basis for calculation. Throughout we refer to the function $\Phi(r)$ as the "potential" and $m(r)$ [$\neq M(r)$] as the effective gravitational mass. A central point of this analysis is the fact that this potential can be obtained without knowledge of $m(r)$. It is immediately clear from (1) that all geodesic orbits are stably planar (say $\theta = \pi/2$) and have two constants of motion, the "energy" $\gamma = e^{2\Phi(r)}\dot{t}$ and "angular momentum" $l = r^2\dot{\phi}$ [6]. In the timelike case then

$$\dot{r}^2 f(r) + \mathcal{V}(r) = \gamma^2, \tag{2}$$

where

$$f(r) = \frac{e^{2\Phi(r)}}{1 - \frac{2m(r)}{r}}, \tag{3}$$

and

$$\mathcal{V}(r) = e^{2\Phi(r)}\left(1 + \frac{l^2}{r^2}\right). \tag{4}$$

Setting $\dot{r} = \ddot{r} = 0$, $r > 0$, it follows from the timelike geodesic equations that [7]

$$\gamma = \frac{e^{\Phi}}{\sqrt{1 - r\Phi'}} \tag{5}$$

and

$$l = \frac{r\sqrt{r\Phi'}}{\sqrt{1 - r\Phi'}}, \tag{6}$$

where the time orientation has been chosen so that $\gamma > 0$ and all particles are assumed to rotate in the same sense with ϕ chosen so that $l > 0$. Note that the existence of these circular orbits requires

$$0 < r\Phi' < 1. \tag{7}$$

Next, we require that the timelike circular geodesics be stable. Let r_0 be a circular orbit and consider $r = r_0 + \delta$, where $\delta \ll r_0$. Taking expansions of $\mathcal{V}(r)$ and $f(r)$ about $r = r_0$, it follows from (2) that

$$\ddot{\delta} + \frac{\mathcal{V}''(r_0)}{2f(r_0)}\delta = 0, \tag{8}$$

so that $\mathcal{V}''(r_0) > 0$ for stability. [The requirement $\mathcal{V}'(r_0) = 0$ merely reproduces (6)]. From (4) then

$$3\Phi' + r\Phi'' > 2r(\Phi')^2 \tag{9}$$

for stable circular orbits, a refinement of the Newtonian condition $3\Phi' + r\Phi'' > 0$ for conservative central fields.

Under the assumption that information travels to us along null geodesics, it follows, without further assumption, that [8]

$$1 + z \equiv \frac{\lambda_o}{\lambda_e} = \frac{(u_\alpha k^\alpha)_e}{(u_\alpha k^\alpha)_o}, \tag{10}$$

where λ is the wavelength, e stands for the emitter, o for the observer, u_α is the timelike four tangent, and k^α is tangent to the null geodesic (N) connecting e and o. The emitter is assumed to be on a stable circular timelike geodesic in (1). Along N, define

$$b \equiv \frac{l_N}{\gamma_N}. \tag{11}$$

The constant γ_N is positive by construction but l_N is both positive and negative. The observer is taken to be a static observer at infinity ($u_o^\alpha = e^{-\Phi(\infty)}\delta_t^\alpha$). Specifically, we assume that $\Phi(\infty) \to C$ where C is a finite constant which we can set to zero without loss in generality. [The "fitting" problem associated with the assumption that $\Phi(\infty)$ is finite is discussed briefly below.] Then b represents the impact parameter at infinity. That is, $|b|$ gives the observed radial distance on either side of the observed center ($b = 0$). The construction of a mapping $b(r)$ (the mapping between the observer and coordinate planes) is an important part of this analysis. Evaluation of (10) now gives

$$1 + z_\epsilon = \frac{1}{\sqrt{1 - r\Phi'}}\left(\frac{1}{e^\Phi} - \frac{\sqrt{r\Phi'}\epsilon|b|}{r}\right), \tag{12}$$

where $\epsilon = \pm 1$. Rather than (12), we consider shifts on either side of the central value ($b = 0$) [9]

$$1 + z_c = \frac{1}{e^\Phi\sqrt{1 - r\Phi'}}. \tag{13}$$

Defining

$$Z \equiv z_+ - z_c = z_c - z_-, \tag{14}$$

we have

$$Z^2 = \frac{\Phi' b^2}{r(1 - r\Phi')}. \tag{15}$$

We now construct the mapping $b(r)$. At fixed b (that is, at a fixed offset from the observed center of the galaxy), choose the maximum observed value of Z. From (15) it follows that, if $\Phi'/[r(1 - r\Phi')]$ is monotone decreasing with increasing r, then the maximum observed value of Z corresponds to the minimum value of r along N. This minimum value follows from the null geodesic equation and is given by

$$b^2 = \frac{r^2}{e^{2\Phi}}. \tag{16}$$

The monotone requirement gives us

$$\Phi' > r\Phi'' + 2r(\Phi')^2. \tag{17}$$

With this restriction the mapping $b(r)$ is given by (16). A derivation of (15) with (16) assumed and without the considerations leading to (17) has been given by Nucamendi, Salgado, and Sudarsky [4].

Observations of galactic rotation curves are reported by way of the "optical convention"

$$v \equiv \frac{\lambda_o - \lambda_e}{\lambda_e}, \tag{18}$$

so that

$$Z = v(b) - v(b = 0) \tag{19}$$

with Z given by (15) subject to the mapping (16). It is important to note that no "velocity" has entered the procedure. Whereas it has become customary to decompose galactic potentials into various parts [10–12], such decompositions insert further assumptions. We continue here with the full function Φ taken as given directly from the observations assuming only that v is corrected for all systematic effects and reflects an intrinsic property of the galaxy alone.

We now seek information on the function m. On the basis of the Lovelock theorem [13], we know that properties of the Einstein tensor are of central importance (even without invoking Einstein's equations). Indeed, for spaces of the form (1), the entire structure of the space can be specified by a single "anisotropy" function

$$\mathcal{H} \equiv G_\theta^\theta - G_r^r, \tag{20}$$

where G_α^β is the Einstein tensor. It follows that [14]

$$m = \frac{\int b e^{\int a dr} dr + C}{e^{\int a dr}}, \tag{21}$$

where

$$a \equiv \frac{2r^2[\Phi'' + (\Phi')^2] - 3r\Phi' - 3}{r(r\Phi' + 1)}, \tag{22}$$

and

$$b \equiv \frac{r\{r[\Phi'' + (\Phi')^2 - \mathcal{H}] - \Phi'\}}{r\Phi' + 1}, \tag{23}$$

with C a constant. Under the assumption of spatial isotropy ($\mathcal{H} = 0$) m is determined, up to quadrature, knowing Φ' and Φ''. Moreover, any function m can be generated by a suitable choice for $\mathcal{H}$. However, m and $\mathcal{H}$ cannot be determined simultaneously without further assumptions. Whereas it is a straightforward matter to specify $\mathcal{H}$ via a specific decomposition of the energy-momentum tensor (and many recent papers do exactly this), such a procedure is not unique in the sense that the same function $\mathcal{H}$ is derivable from inequivalent decompositions. We proceed here in a different way by invoking an assumption based on a Newtonian limit.

To place the foregoing analysis in Newtonian terms, we introduce a potential $\tilde{\Phi}$ defined by

$$\nabla^2\tilde{\Phi} \equiv -R_t^t, \tag{24}$$

where R_t^t is the time component of the Ricci tensor of (1) [15]. In the usual way, we find that $\tilde{\Phi}$ satisfies

$$\tilde{\Phi}' = \frac{m}{r^2} - \frac{\Lambda r}{3} + \frac{3}{r^2}\int_0^r \left(r\Phi' - \frac{m}{r}\right) dr + \frac{1}{r^2}\int_0^r r^2\mathcal{H}\, dr, \tag{25}$$

where we have set $r \gg 2m$ and introduced the cosmological constant Λ. The dynamical mass is defined by $M \equiv r^2\tilde{\Phi}'$ and the balance of Newtonian forces for circular motion gives

$$M = rV^2, \tag{26}$$

where V is the orbital speed. If the frequency shift is assumed to be due to the nonrelativistic Doppler effect, then $V = v$ and, since v is known, $\tilde{\Phi}'$ is known and we now have two equations, (25) and (21), from which we determine a solution (m,$\mathcal{H}$). Note that no decomposition of any energy-momentum tensor has been used.

Let us now define the energy density $8\pi\rho \equiv -G_t^t$ so that

$$m = \int_0^r 4\pi r^2 \rho\, dr \tag{27}$$

just as in Newtonian mechanics (though there ρ stands for the mass density). Equation (25) now provides the link between the dynamical mass M and the density ρ. The traditional "missing mass" problem derives from the fact that, since v^2 levels off at larger values of r for many galactic halos, assuming $V = v$ then according to (26) M must continue to grow in that region like r. Assuming $M = m$, it then follows from (27) that there must be unseen material if the inclusion of all observed contributions to ρ does not produce this continued growth. However, $M \neq m$, and it is possible that the observed contributions to ρ are compatible with m while M continues to grow like r according to (25). In this sense, there would be no mass missing at all. Mass should be considered missing when all observed contributions to ρ are incompatible with m, not M, and such an incompatibility could arise whether or not v^2 levels off.

The fitting problem involves the smooth junction of (1) at a finite value of r (say R) onto an external field in which we can set $\Phi(\infty) = 0$. This boundary condition is required by the definition of b and by the fact that the rotation curves are considered intrinsic (corrected for all other effects). Geometrically this junction is examined by way of the Darmois-Israel conditions [16]. The smooth junction of metrics of the form (1) requires only the continuity of m and Φ', assuming the continuity of r, θ, and ϕ, so that G_r^r, but not G_θ^θ nor G_t^t, is necessarily continuous at R. In general relativity, if the external field is taken as vacuum then in a suitable gauge $e^{2\Phi} = 1 - 2m(R)/r - \Lambda r^2/3$ and so $\Phi(\infty) = 0$ only for $\Lambda = 0$.

In summary, assuming only that emission occurs from stable timelike circular geodesic orbits in a static spherically symmetric metric with information propagated to us along null geodesics, it has been shown that galactic potentials follow directly from the observed rotation curves via the relation $Z^2 = v^2$ where, subject to the mapping (16), Z^2 is given by (15). Neither the gravitational mass m nor any metric theory of gravity enters this determination of the potential Φ but Φ must satisfy (7), (9), and (17). Next, in terms of a single anisotropy function [$\mathcal{H}$ given by (20)] m follows directly from Φ. Both m and $\mathcal{H}$ can be solved, without using any specific decomposition of the energy-momentum tensor, by constructing a Newtonian limit and by now assuming that the frequency shift is due to the nonrelativistic Doppler effect. This procedure naturally defines a dynamical mass $M \neq m$ with respect to which the missing mass problem is usually defined. With Φ, $\mathcal{H}$, and m determined, the dark matter, should it be required, is highly constrained, but not identified since the same function $\mathcal{H}$ is derivable from inequivalent decompositions of the energy-momentum tensor.

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*Electronic address: lake@astro.queensu.ca

[1] See, for example, L. Bergstrom, Rep. Prog. Phys. **63**, 793 (2000).

[2] An influential early work is S. Faber and J. Gallagher, Annu. Rev. Astron. Astrophys. **17**, 135 (1979). For a recent review, see Y. Sofue and V. Rubin, Annu. Rev. Astron. Astrophys. **39**, 137 (2001).

[3] See, for example, R. Sanders and S. McGaugh, Annu. Rev. Astron. Astrophys. **40**, 263 (2002).

[4] There are a large number of recent works is the same general spirit as this paper. See, for example, U. Nucamendi, M. Salgado, and D. Sudarsky, Phys. Rev. D **63**, 125016 (2001); L. Cabral-Rosetti, T. Matos, D. Nuñez, and Roberto A. Sussman, Classical Quantum Gravity **19**, 3603 (2002); F. E. Schunck, astro-ph/9802258; E. W. Mielke and F. E. Schunck, Phys. Rev. D **66**, 023503 (2002); S. Bharadwaj and S. Kar, Phys. Rev. D **68**, 023516 (2003).

[5] We use geometrical units throughout, a signature of $+2$ and the summation convention.

[6] Here an overdot ($\dot{}$) signifies differentiation with respect to an affine parameter, the proper time in the timelike case.

[7] Henceforth we do not explicitly write out functions of r [e.g., $\Phi \equiv \Phi(r)$], and we use $' \equiv d/dr$.

[8] For a careful derivation, see E. Schrödinger, *Expanding Universes* (Cambridge University Press, Cambridge, England, 1956).

[9] It is instructive to start at the center ($b = 0$) and examine (13). Suppose that $1 + z_c = 1$ (which, it would appear, is consistent with the observations) so that from (13) $e^{2\Phi} = 1 - cr^2$, where c is the constant of integration. Both (7) and (9) require $c < 0$, and so the required potential is of the anti–de Sitter form, though there is no suggestion that c is universal. This already makes clear that there is a fitting problem in the sense that if $\Phi(\infty) = 0$ then the fundamental character of $\Phi(r)$ changes in the intervening region.

[10] See, for example, P. Salucci and M. Persic, ASP Conf. Series **117**, 1 (1997); E. Battaner and E. Florido, Fund. Cosmic Phys. **21**, 1 (2000). On the basis of a large number rotation curves, Persic and Salucci have suggested a "universal" rotation curve intrinsic to galactic halos. Although any such "universal" form can be expected to see exceptions (for example, A. Bosma, ASP Conf. Series **182**, 339 (1998)], we use it here as a demonstration to show how observations determine Φ. The curve is of the form $v(b)^2 = (\alpha b^2)/(b^2 + \beta)$, where α and β are positive constants characteristic of a particular galaxy.

[11] It follows from (15) and (16) then that the potential is governed by

$$\frac{\Phi' r}{e^{2\Phi}(1 - r\Phi')} = \frac{\alpha r^2}{r^2 + \beta e^{2\Phi}}. \qquad (28)$$

Whereas for small r it follows that we can set $\Phi(0) = 0$ (so again $1 + z_c = 1$), the full solution to (28) is

$$e^{2\Phi} = \frac{r^2}{\alpha r^2 \mathcal{W}(y) - \beta}, \qquad (29)$$

where $\mathcal{W}$ is the Lambert W function [$\mathcal{W}$ is defined by the condition $\mathcal{W}(x)e^{\mathcal{W}(x)} = x$], $y = -[(de^{\beta/(\alpha r^2)})/(\alpha^2 r^2)]$ and d is a constant of integration ($\neq 0$).

[12] The potential (29) exactly reproduces the "universal" rotation curve without reference to a background source or even a theory of gravity. See, for example, Schunck [4] for a discussion of scalar fields and the more recent papers T. Matos and F. Guzman, Classical Quantum Gravity **17**, L9 (2000); T. Matos, F. Guzman, and D. Nunez, Phys. Rev. D **62**, 061301 (2000); M. Alcubierre, F. Guzman, T. Matos, D. Nunez, L. Urena, and P. Wiederhold, Classical Quantum Gravity **19**, 5017 (2002). Both (7) and (9) require $d < 0$ and (17) requires $d < -\alpha\beta$. At large r (in a suitable gauge), the solution (29) takes the form $e^{2\Phi} = 1 - [(\alpha^2 r^2)/d] + \mathcal{O}(1/r^2)$. Alternate full rotation curves, not just a "halo" component, can be considered and Φ solved in exactly the same way. In general, one would expect recourse to numerical procedures, but in all cases Φ must satisfy conditions (7), (9), and (17).

[13] D. Lovelock, J. Math. Phys. (N.Y.) **13**, 874 (1972).

[14] See K. Lake, Phys. Rev. D **67**, 104015 (2003) for the case $\mathcal{H} = 0$. Somewhat simpler forms of the procedure described there have been given recently by D. Martin and M. Visser, gr-qc/0306109, and by Jan Gogolin (private communication). Here we continue to use Φ' and Φ'' because of conditions (7), (9), and (17).

[15] Defining $8\pi p_R \equiv G^r_r$, we have

$$r\Phi' = \frac{8\pi p_R r^3 + 2m}{2(r - 2m)}. \qquad (30)$$

First consider $r \gg 2m$ (as in, for example, Mielke and Schunck [4]). Then

$$r\Phi' \simeq 4\pi p_R r^2 + \frac{m}{r}. \qquad (31)$$

By inserting the exact form of p_R from (30), (31) reduces to

$$\Phi' \simeq 0. \qquad (32)$$

A different approximation has been considered by Schunck [4] [see Schunck's Eq. (19)]. This approximation is equivalent to

$$e^{2\Phi} \simeq 1 - \frac{2m}{r}. \qquad (33)$$

Neither of the approximations (32) nor (33) produce a workable Newtonian limit for our considerations here.

[16] P. Musgrave and K. Lake, Classical Quantum Gravity **13**, 1885 (1996).

[17] This is a package which runs within MAPLE. It is entirely distinct from packages distributed with MAPLE and must be obtained independently. The GRTensorII software and documentation is distributed freely on the World Wide Web from the address http://grtensor.org