Warm Cascades and Anomalous Scaling in a Diffusion Model of Turbulence

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A phenomenological turbulence model in which the energy spectrum obeys a nonlinear diffusion equation is analyzed. The general steady state contains a nonlinear mixture of the constant-flux Kolmogorov and fluxless thermodynamic components. Such "warm cascade" solutions describe a bottleneck phenomenon of spectrum stagnation near the dissipative scale. Transient self-similar solutions describing a finite-time formation of steady cascades are analyzed and found to exhibit nontrivial scaling behavior.

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INTRODUCTION

A theoretical understanding of the statistics of hydrodynamic turbulence and the origin of Kolmogorov's 5/3 spectrum, postulated from dimensional considerations in 1941, is one of the outstanding problems of classical physics which continues to resist modern efforts at solution. The difficulty lies in the strong nonlinearity of the governing Navier-Stokes equations which leads to an unclosed hierarchy of equations which must be solved to obtain the moments of the velocity distribution. Given the seeming intractability of this so-called "closure problem," models have come to play an important role in understanding the physics of turbulence over the years. A model of a physical system aims to encapsulate certain properties of interest while disregarding others in the hope that understanding can be translated back to the original system from the simpler one. In this Letter we are interested in a class of models which could be called spectral diffusion models. These describe the time evolution of the spectral energy density, $E(k, t)$, in terms of a partial differential equation by making a diffusion approximation to the energy transport process in the wave number or k-space representation. One of the first such models was studied by Kovasznay [1] who proposed a first order equation based on a dimensionally consistent phenomenological expression for the energy flux which admits the Kolmogorov $k^{-5/3}$ spectrum as a stationary solution. Later, Leith [2] proposed a second order equation which allowed for the possibility of a stationary thermodynamic equilibrium spectrum in addition to the Kolmogorov spectrum. Variations on Leith's model have been analyzed extensively since they facilitate the study of the dynamics which govern the establishment and decay of the Kolmogorov spectrum without the need to analyze the nonlocal energy transfer process inherent in the Navier-Stokes equations. It has been shown by Besnard *et al.* [3] that a broad range of spectral diffusion models exhibit self-similar decay spectra consistent with the 5/3 law.

In this Letter, we focus on one such model equation which is very close to Leith's original [4]:

$$\frac{\partial E}{\partial t} = \frac{1}{8}\frac{\partial}{\partial k}\left[k^{11/2}E^{1/2}\frac{\partial}{\partial k}(E/k^2)\right] + f - \nu k^2 E, \quad (1)$$

where t is time, k is the absolute value of the wave number, ν is the kinematic viscosity coefficient and f is an external forcing. $E(k, t)$ is normalized so that the kinetic energy density is $\int E\,dk$. We select this particular equation from the family of equations studied in [3] on the basis that, in the absence of the forcing and dissipation terms, it admits the correct thermodynamic spectrum, $E(k) = Q^{2/3}k^2$, in addition to the Kolmogorov spectrum, $E(k) = P^{2/3}k^{-5/3}$, as particular steady-state solutions. In these spectra, P and Q are constants. In what follows we note that the general steady state is actually a nonlinear combination of both thermodynamic and Kolmogorov components. We find that in the presence of forcing and damping, assuming that an inertial range interval exists, these mixed spectra can be relevant if the dissipation is too weak or the high k cutoff too small. We also investigate the nonstationary solutions of (1) in the inviscid case beginning with an initial spectrum compactly supported at low k. We discover the existence of a transient self-similar regime preceeding the breaking of energy conservation (which occurs once the cascade has proceeded far enough to generate a finite flux of energy to $k = \infty$). This regime is interesting because it does not exhibit the scaling inherited from the Kolmogorov spectrum. These results are in line with recent studies [5] of analogous problems in the theory of wave turbulence and may point to a generic phenomenon observed in the dynamics of flux spectra of Kolmogorov type.

STATIONARY SOLUTIONS

First, let us consider steady-state spectra in the inertial range. For $f = \nu = 0$, we have the following general time-independent solution,

$$E = C\,k^2\,(Pk^{-11/2} + Q)^{2/3}, \tag{2}$$

where $C = (24/11)^{2/3} \approx 1.68$ and P and Q are arbitrary constants. For $Q = 0$, this gives the pure Kolmogorov cascade solution, whereas for $P = 0$ this is a pure thermodynamic equilibrium. For the general solution, both the constant flux of energy $P = -\frac{1}{8}k^{11/2}E^{1/2}(E/k^2)_k \neq 0$ and a thermodynamic part ($Q \neq 0$) are present as a nonlinear combination. We refer to solution (2) with finite P and Q as a *warm cascade* to distinguish it from the pure Kolmogorov solution which could be viewed as a *cold cascade*.

Let us suppose that the forcing is compactly supported in a narrow interval around $k = k_0$ and that a large inertial range exists to the right which ends at a very high $k \sim k_d$ where viscosity ν or some other dissipation mechanisms $[f(k) < 0]$ become important. Then, upscale of the forcing there will be a pure thermodynamic solution with $P = 0$ and $Q \neq 0$ because there is no dissipation or forcing assumed to be present near $k = 0$ to absorb or generate a finite energy flux. In the inertial range there will be a constant-flux cascade solution. This solution typically takes the form of a pure Kolmogorov (cold) cascade and extends down to the dissipation range where the energy flux is absorbed. The solution only penetrates a finite distance into the dissipation range and adapts itself until it provides sufficient dissipation to absorb the supplied flux, a point noted implicitly in [3]. The model does not develop structure at arbitrarily high k as it would in the inviscid case. The qualitative features of the steady state are independent of the detailed form chosen for the dissipation. We studied a family of dissipation functions,

$$\nu(k) = \begin{cases} \nu_0(k - k_D)^{\alpha}, & k \geq k_D, \\ 0, & k < k_D, \end{cases} \tag{3}$$

where ν_0, k_D, and α are adjustable parameters. Figure 1 shows the steady-state solutions in the inertial range obtained numerically for several choices of these parameters.

However, if the dissipation is too weak, the solution can penetrate far enough into the dissipation range to reach the maximal wave number which necessarily exists in any numerical solution. If one imposes a zero flux condition at the right end of the computational interval, the energy flux is reflected from the maximal wave number leading to greater values of E in the dissipative range. Such a cascade stagnation acts to enhance the dissipation rate and thereby to adjust it to the energy flux to be absorbed. A similar phenomenon is common in numerical simulations of turbulence and is sometimes called a

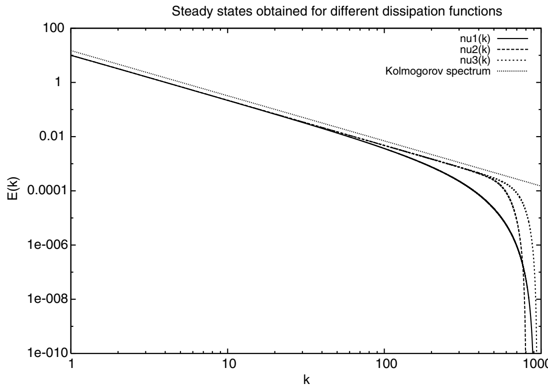

FIG. 1. Numerically computed steady states for several choices of dissipation function (3) : $\nu_1(k)$ has $\nu_0 = 1$, $k_D = 500$ and $\alpha = 2$, $\nu_2(k)$ has $\nu_0 = 4.0 \times 10^{-6}$, $k_D = 500$ and $\alpha = 4$, $\nu_3(k)$ has $\nu_0 = 1.0 \times 10^{-2}$, $k_D = 0$ and $\alpha = 2$. The Kolmogorov spectrum is also shown for comparison but shifted slightly for clarity.

bottleneck. Figure 2 shows a numerically obtained steady state for a dissipation function, having $\nu_0 = 1.0 \times 10^{-5}$, $k_D = 500$, and $\alpha = 2$. The bottleneck-like phenomenon is clearly seen as an energy "pile up" over the cold cascade solution near the dissipative scale. In our model, the bottleneck phenomenon is described by the *warm* cascade solutions; in particular, the theoretical curve in Fig. 2 is computed by taking $P \approx 14.5$, $Q \approx 1.5 \times 10^{-9}$ in Eq. (2). We should note, however, that this bottleneck in the Leith model differs from the bottleneck in the Navier-Stokes equations as explained by Falkovich [6] in several respects. In particular, the usual bottleneck is intrinsically nonlocal and, according to [6], can occur without a high k cutoff. On the other hand, warm cascades in the Leith model are necessarily local and require a k-space cutoff. Nevertheless the phenomenon of an increase in turbulence level at small scales to compensate for a frustrated energy transfer mechanism is common to both.

Some insight about the qualitative behavior of the system can be gained from considering stationary

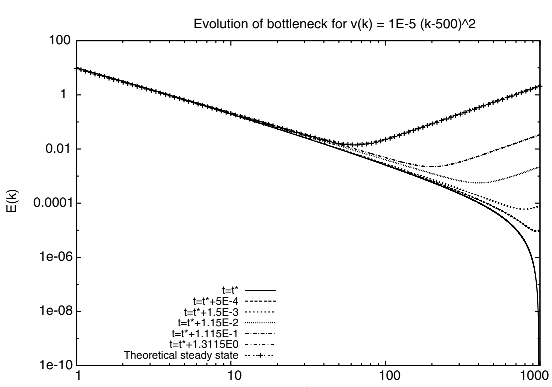

FIG. 2. Numerical evolution of a bottleneck for dissipation function (3) with $\nu_0 = 1.0 \times 10^{-5}$, $k_D = 500$ and $\alpha = 2$. The resulting steady state is well described by the solution (2).

solutions (2) in an inertial range $k_1 < k < k_2$ and fixing the spectrum at its boundaries, $E(k_1) = E_1, E(k_2) = E_2$. These boundary conditions might roughly model the forcing and the dissipation effects outside of the inertial range. This gives

$$P = [(E_2/Ck_2^2)^{3/2} - (E_1/Ck_1^2)^{3/2}]/(k_2^{-11/2} - k_1^{-11/2}), \tag{4}$$

$$Q = [k_2^{5/2}(E_2/C)^{3/2} - k_1^{5/2}(E_1/C)^{3/2}]/(k_2^{11/2} - k_1^{11/2}). \tag{5}$$

Thus, the sign of P is opposite to the sign of $(E_2/E_1 - k_2^2/k_1^2)$ and can be either positive or negative depending on the spectrum steepness with the thermodynamic k^2 solution being a borderline case for which $P = 0$. The constant Q can also be either positive or negative with the Kolmogorov $-5/3$ being the borderline slope. It is convenient to think of the $Q < 0$ solutions as negative temperature states although it is unlikely that such states could arise from physically reasonable forcing and dissipation profiles.

NONSTATIONARY SOLUTIONS

So far we concentrated on the stationary solutions but how do these solutions form? We consider from now on the inviscid case. The Kolmogorov $-5/3$ spectrum is of a *finite capacity* type in that it contains only a finite amount of energy at the high k end. If we take as an initial condition a spectrum which is compactly supported, an infinitely remote dissipative scale should therefore be reached in a finite time. It is well known (see, for example, [7] and the references therein) that the solutions of nonlinear diffusion equations often have the property of remaining compactly supported during the time evolution if the initial data is compactly supported. This turns out to be the case here. The solution has a "sharp" nonlinear front at $k = k_*(t)$ and this front accelerates explosively, reaching $k = \infty$ at a finite singular time which we shall denote by t_*. Let us look for a self-similar solution taking the following form:

$$E = (t_* - t)^a F(\eta); \qquad \eta = k/k_*, \qquad k_* = c(t_* - t)^b, \tag{6}$$

where a, b, and c are constants. Clearly, b must be negative since we require that $k_* \to \infty$ as $t \to t_*$. Substituting (6) into (1) with $f = \nu = 0$ we find that the t dependence drops out of the equation if $a = -2 - 3b$. We then have the following equation for F,

$$(3b + 2)F + b\eta F' = \frac{c^{3/2}}{8}[\eta^{11/2}F^{1/2}(F/k^2)']', \tag{7}$$

where prime means differentiation with respect to η. Equation (7) defines a one-parameter family of self-similar solutions. The solution near the front tip can be found by expanding F in series with respect to small $(1 - \eta)$; in the leading order we have

$$F = \frac{16b^2}{c^3}(1 - \eta)^2, \tag{8}$$

which gives for the spectrum

$$E = \frac{16b^2}{k_*^3(t_* - t)^2}\left(1 - \frac{k}{k_*}\right)^2. \tag{9}$$

We look for solutions which behave like a power law far behind the front. That is, $E \sim k^{-x}$ as $k \to 0$. The relations (6) then imply that $x = -a/b$. The pure Kolmogorov spectrum, $x = 5/3$, therefore requires $b = -3/2$, corresponding to what one might consider to be normal scaling in the wake of the front.

We performed numerical simulations of both the forced and decaying solutions of Eq. (1) with compact initial data to check the development of a self-similar front with a tip of the form (8) and to determine which value of b is selected. The results for the forced case are shown in Fig. 3. Corresponding results for the decaying case can be found in [8]. The scaling parameter, b, and the singular time, t_*, are most conveniently extracted from the relation

$$k_*\left(\frac{dk_*}{dt}\right)^{-1} = -\frac{1}{b}(t_* - t), \tag{10}$$

which allows one to calculate b and t_* from a linear fit of the data near $t = t_*$ as shown in Fig. 4. We find that $t_* = 0.0799$ and $b = -1.748$ which corresponds to a significantly steeper than Kolmogorov slope, $x = 1.856$. The singular time, t_* depends on the choice of initial conditions but the anomalous scaling exponent does not. In particular, we found that the same value of b is obtained in the case of decaying turbulence. Such anomalous scaling behavior whereby the exponent of the solution in the

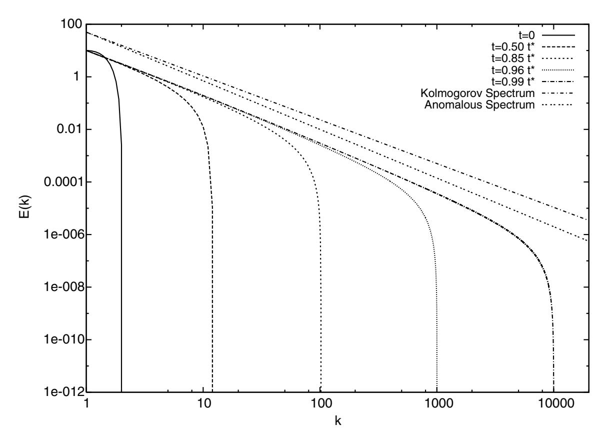

FIG. 3. Forced time-dependent solutions beginning from compact initial data showing development of self-similar front with power law wake. The Kolmogorov and transient spectra are also shown, offset for clarity.

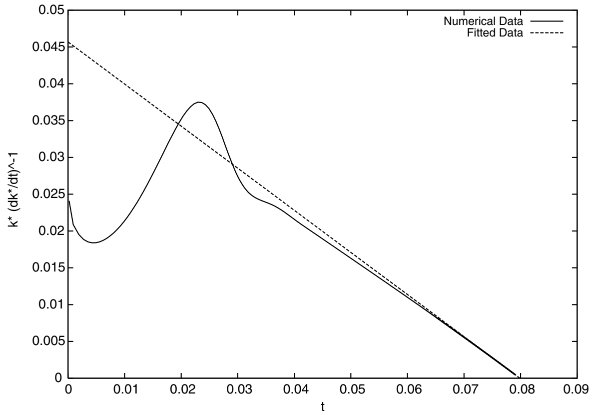

FIG. 4. Calculation of the asymptotic scaling properties of the self-similar solution for the forced case. The fitted line has slope $1/b = -0.572$ and t intercept $t^* = 0.0799$.

wake of the nonlinear front is larger than the Kolmogorov value has been observed before. Examples include MHD wave turbulence [9] and weak turbulence with local interactions [5].

For the model (1) the origin of the anomaly can be traced to the question of existence of a solution of the similarity Eq. (7) which has the correct behavior *both* at the front tip, $\eta \rightarrow 1$, and in the wake, $\eta \rightarrow 0$. Written in terms of x rather than b and rescaled to get rid of the constant c, the similarity equation is

$$\frac{2}{x-3}\left(\eta \frac{dF}{d\eta} + xF\right) = \frac{d}{d\eta}\left[\eta^{11/2}\sqrt{F}\frac{d}{d\eta}(\eta^{-2}F)\right]. \quad (11)$$

We require that this equation have a solution which behaves as η^{-x} as $\eta \rightarrow 0$ and behaves as $(1-\eta)^2$ as $\eta \rightarrow 1$. Such a solution is not typical and actually exists only for one value of x. In particular, such a solution does not exist for $x = 5/3$, the Kolmogorov value. The structure of the problem can be studied by introducing a new independent variable, $s = \log \eta$ and a pair of dependent variables, $f(s)$, $g(s)$ defined by

$$F = \frac{1}{25}\,\eta^{-3} f^2, \qquad \frac{dF}{d\eta} = \frac{3}{25}\,\eta^{-4} fg. \quad (12)$$

Equation (11) is equivalent to the following autonomous first order system:

$$\frac{df}{ds} = \frac{3}{2}(f+g), \quad (13)$$

$$f\frac{dg}{ds} = \frac{1}{3}\left[5f^2 + 6fg - 9g^2 + \frac{10}{x-3}(3f + xg)\right].$$

The associated dynamical system has three equilibria $P1 = (0,0)$, $P2 = [0, 10/3(x-3)]$, and $P3 = (1,-1)$. Note that $P1$ and $P2$ are singular points of the original equations, (13). The point P1 can be shown to correspond to the wake and $P2$ to the tip. The required solution exists when the unstable manifold of P1 intersects the stable manifold of $P2$. Numerical investigation of the phase plane shows that this happens only for $x \approx 1.85$. For details, see [8].

In practice, any model should include dissipation so that this self-similar solution above will be valid only until the front tip meets the dissipation scale. After this, the transient slope gets replaced in the inertial range by the stationary cascade solution, with or without bottleneck depending on the dissipation, as discussed above.

CONCLUSION

We conclude with some brief comments about the possible wider applicability of these ideas and about which features might be model dependent. First, we expect that the presence of a sharp front in the Leith model is a property of the locality of the energy transfer. Although this feature would surely disappear once nonlocal inteactions are taken into account, the notion of finite energy capacity remains valid. Therefore the transient regime discussed here might also be present in Navier-Stokes turbulence. Second, one of the potential practical uses of spectral diffusion models lies in coupling them to more complicated models, computer simulations, for example, in order to provide a description of small scale turbulence. Our analysis of the warm cascades which can develop if the dissipation is insufficient should caution us that such applications should be constructed with care.

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