Space-Filling Bearings in Three Dimensions

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We present the first space-filling bearing in three dimensions. It is shown that a packing which contains only loops of an even number of spheres can be constructed in a self-similar way and that it can act as a three-dimensional bearing in which spheres can rotate without slip and with negligible torsion friction.

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Space-filling bearings have been introduced in several contexts, such as in explaining the so-called seismic gaps [1,2] of geological faults in which two tectonic plates slide against each other with a friction much less than the expected one, without production of earthquakes or of any significant heat. Space-filling bearings have also been used as toy models for turbulence and can also be used in mechanical devices [3]. Two-dimensional space-filling bearings have been shown to exist and a discrete infinity of realizations has been constructed [4,5]. The remaining question still open is: Do they also exist in three dimensions? This question is of fundamental importance to the physical applications.

In this Letter, we will report the discovery of a self-similar space-filling bearing in which an arbitrary chosen sphere can rotate around any axis and all the other spheres rotate accordingly without any sliding and with negligible torsion friction.

In two dimensions, different classes of space-filling bearings of disks have been constructed in Refs. [4,5] by requiring the loops to have an *even* number of disks, since in two dimensions this is obviously a necessary and sufficient condition for disks to be able to rotate without any slip. Successive disks must rotate, in alternation, clockwise and counterclockwise in order to avoid frustration.

The situation in three dimensions is different from two dimensions in two ways: The axes of rotation need not be parallel, and the centers of spheres in a loop may not all lie in the same plane. As a result, even in an *isolated* odd loop spheres could rotate without friction. But, as we will see, in the packings with an infinite number of interconnecting loops, we could construct unfrustrated configurations of rotating spheres when all loops have an even number of spheres. Such a packing is *bichromatic*; i.e., one can color the spheres using only two different colors in such a way that no spheres of the same color touch each other, as shown in Fig. 1.

No three-dimensional space-filling bearing has been known until now. The classical Apollonian packing is space filling and self-similar but not a bearing since at least *five* colors are needed to assure different colors at each contact. This packing can be constructed in different ways [6,7]. By generalizing the inversion technique used in Ref. [6] to other *Platonic solids* than the tetrahedron (the base of 3D Apollonian packing), we were able to construct five new packings including a bichromatic one. Details on the construction algorithm and on the complete set of new configurations will be published elsewhere [8]. We give here only a qualitative description of this technique for the bichromatic packing.

Let us consider filling a sphere of unit radius. The filling procedure is initialized by placing seven initial spheres on the vertices and the center of a regular *octahedron* inside the unit sphere, so that the spheres on the vertices do not touch each other but touch the unit sphere and the one in the center. Further spheres are inserted by inversion [9] such that this topology can be preserved on all scales, imposing that all spheres are on vertices or centers of (deformed) octahedra.

The inversions are made iteratively with respect to nine *inversion spheres* [10]: One inversion sphere is concentric with the unit sphere, and is perpendicular to the six initial spheres on the vertices of the octahedron [11]. Inversion

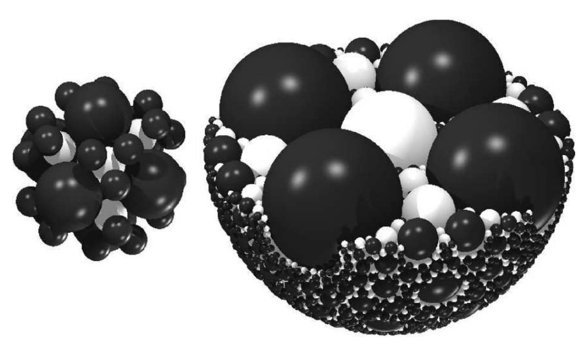

FIG. 1. Bichromatic packing of spheres with octahedral symmetry. No two spheres of the same color touch each other. The image on the left shows the initial configuration and the first generation of inserted spheres.

around it, therefore, leaves the vertex spheres invariant, and maps the unit sphere onto the central sphere. The eight other inversion spheres are associated with each face of the octahedron; that is, they are perpendicular to the unit sphere and to the three spheres on the vertices of that face. The inversion around each of these inversion spheres maps the unit sphere and the spheres of the corresponding face into themselves (that is, it gives no new sphere), and the other four initial spheres into four new spheres within the space between the face and the unit sphere.

Figure 2 shows the plane cut through the centers of the unit sphere and four initial spheres. Dashed circles are cuts of the inversion spheres. Sphere S:0, 1, … is mapped, by the inversion sphere shown by a thick dashed line, onto sphere S'. The inversion around this sphere gives no new images of spheres 1 and 2. In the first iteration, we make all possible inversions which give new and smaller spheres. In the next iterations, the newly generated spheres are mapped to smaller spheres. For example, sphere $0'$ is mapped (by the central inversion sphere) onto $0''$. In this way, the remaining empty space is filled in the limit of infinite iterations while the bichromatic topology of the contacts is preserved.

Using this algorithm, the configuration of initial spheres which gives a bichromatic packing is unique. Strictly speaking, it is shown that the only value for the radii of the initial spheres, which leads to the bichromatic packing without the (partial) overlapping of generated spheres, is $(\sqrt{3}-1)/2$ and only using an octahedral base. The fractal dimension of the packing has been computed using the same method as in Ref. [4]; it is 2.59 and considerably higher than that of Apollonian packing in 3D, which is 2.47 [6].

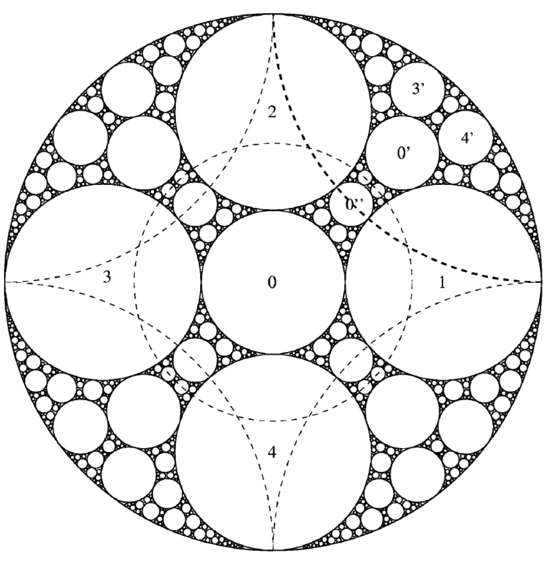

FIG. 2. Plane cut through the center of the unit sphere and four initial spheres. Dashed circles are cuts of the inversion spheres. Sphere S is mapped, by the inversion sphere shown by a thick dashed line, onto sphere S'. $0'$ is mapped by the central inversion sphere onto $0''$.

The image on the left of Fig. 1 shows the initial configuration and the first generation of inserted spheres; the one on the right shows the resulting packing containing all the spheres with radii greater than 2^{-7}. The sphere at the center and the external hull are white and those on the vertices of the octahedron are black. Since the spheres on the vertices touch only the external hull and the central sphere, and since this topology is preserved by construction, no two spheres of the same color touch each other.

This implies that every loop of spheres in this packing contains an even number of spheres. We now show that this is a sufficient condition for the spheres in contact to rotate without slip, or even torsion friction.

Consider a loop of n spheres as seen schematically for $n = 4$ in Fig. 3. The no-slip condition implies that each pair of touching spheres has the same tangent velocities $\vec{v}$ at their contact point. The condition for the contact between the first and the second sphere can be written as

$$\begin{aligned} \vec{v}_1 = \vec{v}_2 &\Rightarrow R_1\hat{r}_{12} \times \vec{\omega}_1 = -R_2\hat{r}_{12} \times \vec{\omega}_2 \\ &\Rightarrow (R_1\vec{\omega}_1 + R_2\vec{\omega}_2) \times \hat{r}_{12} = 0, \end{aligned} \tag{1}$$

where R_1, R_2, $\vec{\omega}_1$, and $\vec{\omega}_2$ are the radii and the vectorial angular velocities of the first and the second sphere, respectively. $\hat{r}_{12}$ is the unit vector in the direction connecting the centers of the first and the second sphere. From Eq. (1), the vector $(R_1\vec{\omega}_1 + R_2\vec{\omega}_2)$ should be parallel to $\hat{r}_{12}$:

$$R_2\vec{\omega}_2 = -R_1\vec{\omega}_1 - \alpha_{12}\hat{r}_{12}, \tag{2}$$

where α_{12} is an arbitrary parameter. Equation (2) is a connection between the rotation vectors $\vec{\omega}_1$ and $\vec{\omega}_2$ of the

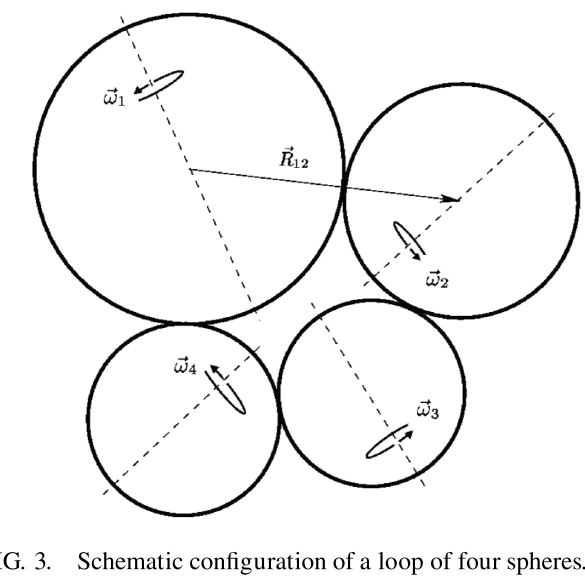

FIG. 3. Schematic configuration of a loop of four spheres. The spheres rotate without slip. The centers of spheres need not be in the same plane, although $\vec{\omega}_1$, $\vec{\omega}_2$, and $\vec{R}_{12}$ are coplanar.

two spheres in contact. Similarly, for the third sphere in contact with the second, we have

$$R_3\vec{\omega}_3 = -R_2\vec{\omega}_2 - \alpha_{23}\hat{r}_{23}. \tag{3}$$

Putting Eq. (2) into Eq. (3), we find the relation between the angular velocities of the first and the third sphere:

$$R_3\vec{\omega}_3 = R_1\vec{\omega}_1 + \alpha_{12}\hat{r}_{12} - \alpha_{23}\hat{r}_{23}. \tag{4}$$

In general, we can relate the angular velocities of the first and the jth spheres of an arbitrary chain of spheres in no-slip contacts by

$$R_j\vec{\omega}_j = (-1)^{j-1}R_1\vec{\omega}_1 + \sum_{i=1}^{j-1}(-1)^{j-i}\alpha_{i,i+1}\hat{r}_{i,i+1}. \tag{5}$$

As long as the chain is open, the spheres can rotate without slip with the angular velocities given by Eq. (5) and no restrictions on $\alpha_{i,i+1}$. But, for a loop of n spheres in contact, spheres j and $j+n$ are identical, so that

$$R_1\vec{\omega}_1 = (-1)^n R_1\vec{\omega}_1 + \sum_{i=1}^{n}(-1)^{n-i+1}\alpha_{i,i+1}\hat{r}_{i,i+1}. \tag{6}$$

A similar equation holds for every sphere $j = 1, \ldots, n$ in the loop.

Although for a single loop there are many solutions of Eq. (6), not all will serve our purpose. In a packing, each sphere belongs to a very large number of loops and all loops should be consistent and avoid frustration. In other words, the angular velocity obtained for a sphere as a member of one loop should be the same as being a member of any other loop.

If the loop contains an even number n of spheres, Eq. (6) becomes a relation between the hitherto arbitrary coefficients of connection $\alpha_{i,i+1}$,

$$\sum_{i=1}^{n}(-1)^i\alpha_{i,i+1}\hat{r}_{i,i+1} = 0. \tag{7}$$

Using the fact that the loop is geometrically closed,

$$\sum_{i=1}^{n}(R_i + R_{i+1})\hat{r}_{i,i+1} = 0, \tag{8}$$

a solution for Eq. (7) is

$$\alpha_{i,i+1} = c(-1)^i(R_i + R_{i+1}), \tag{9}$$

where c is an arbitrary constant. Putting this in Eq. (5) yields the angular velocities

$$\vec{\omega}_j = \frac{1}{R_j}(-1)^j(-R_1\vec{\omega}_1 + c\vec{R}_{1j}), \tag{10}$$

where $\vec{R}_{1j}$ is the vector which connects the centers of the first and the jth sphere. As can be seen, the angular velocities depend only on the positions of the spheres, so that the consistency between different loops can be automatically fulfilled providing that the parameter c is the same for every loop of the entire packing.

In Eq. (10), all the angular velocities are calculated from $\vec{\omega}_1$ and c, which can be chosen arbitrarily. ($c = 0$ corresponds to the case when all angular velocities are parallel.)

The no-slip condition (2) then reads

$$R_1\vec{\omega}_1 + R_2\vec{\omega}_2 = c\vec{R}_{12}, \tag{11}$$

so that the vectors $\vec{\omega}_1$, $\vec{\omega}_2$, and $\vec{R}_{12}$ are coplanar (the plane of Fig. 3, containing the two centers and the point of contact A). They are in general not collinear.

We note that the condition of rotation without slip (2) also guarantees that there will be no torsion friction, as long as the three vectors are not collinear. Indeed, the locus of the contact point A on sphere 1 is a circle C_1, perpendicular to $\vec{\omega}_1$. On sphere 2, it is a circle C_2, perpendicular to $\vec{\omega}_2$. The cone tangent to sphere $i = 1, 2$ on circle C_i has an apex S_i, on the axis $\vec{\omega}_i$. The two apices S_i are in the same plane as the sphere centers and on the same line as the contact point A (Fig. 4). The two spheres rolling on each other can therefore be replaced by the two cones rolling on each other, around the line S_1S_2A, which always contains the contact point A. There will be no twisting between cones, thus no torsion friction from one sphere rolling on the other. Only when the two angular velocities and $\vec{R}_{12}$ are collinear can there be some twist of the spheres against each other. (The circles C_i then reduce to the contact point A, there are no tangent cones, and the tangent velocities $v_i = 0$.) This situation is not generic. It is of measure zero and physically irrelevant.

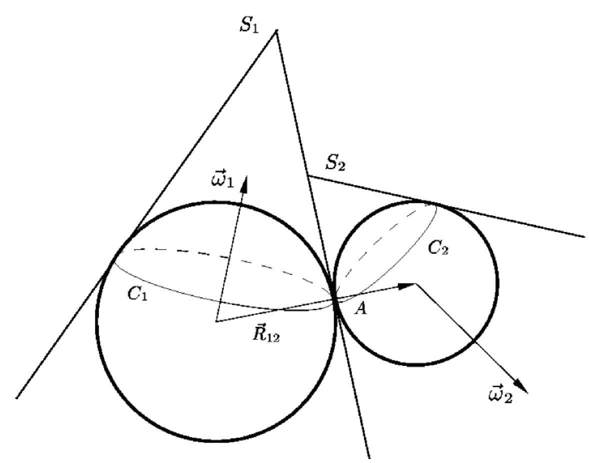

FIG. 4. Two spheres rolling on each other without slip can be represented by two cones rolling on each other around the common line S_1S_2A. Therefore there is no twist of the spheres against each other.

In the case of odd loops, Eq. (6) becomes

$$\sum_{i=1}^{n}(-1)^i \alpha_{i,i+1}\hat{r}_{i,i+1} = 2R_1\vec{\omega}_1. \tag{12}$$

A similar equation holds for every sphere $j = 1, \ldots, n$ in the loop. But, since the coefficients $\alpha_{i,i+1}$ depend then on both geometry and the rotation velocities of the spheres of the loop, consistency between different loops cannot be fulfilled in general and, therefore, a packing containing odd loops cannot be a bearing. A rigorous proof for this, however, is missing. It should be mentioned, however, that an unfrustrated *single* odd loop is possible, but cannot occur in isolation in a packing [12].

The bearing discussed here is very idealized and based on exactly spherical particles with infinite rigidity and, of course, does not exist in this form on all length scales in nature. Nevertheless, it is possible that similar bearings, though with some imperfections, occur in reality. A simulation of two-dimensional shear bands shows the formation of spontaneous rotating bearings in clusters of up to 30 particles [13]. Despite having greater volume, the bearing state is favored because of its low friction. As further evidence, the bichromatic packing presented here is self-similar, which is also observed in the samples of tectonic gouge down to several scales (see Ref. [14]). Interestingly, the measured fractal dimension, 2.60 ± 0.1, agrees with what we obtained in this work.

In summary, we proved the existence of the three-dimensional space-filling sphere bearing by presenting the explicit construction. We have shown that a sufficient condition is that the packing is bichromatic, and we have given an explicit expression (10) for the angular velocity of every sphere of the entire packing, in terms of $\vec{\omega}_1$ and c only. In this way, we support the previous modelization for lubrication between tectonic plates. This result can also be important in mechanics and hydrodynamics.

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