B_K from Quenched QCD with Exact Chiral Symmetry

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We present a calculation of the standard model $\Delta S = 2$ matrix element relevant to indirect CP violation in $K \to \pi\pi$ decays which uses Neuberger's chiral formulation of lattice fermions. The computation is performed in the quenched approximation on a $16^3 \times 32$ lattice that has a lattice spacing $a \sim 0.1$ fm. The resulting bare matrix element is renormalized nonperturbatively. Our main result is $B_K^{\mathrm{RGI}} = 0.87(8)^{+2+14}_{-1-14}$, where the first error is statistical, the second is systematic, and the third is Sharpe's estimate of quenching and flavor-SU(3) breaking uncertainties.

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Ginsparg-Wilson (GW) fermions [1–4], with their exact chiral-flavor symmetry at finite lattice spacing [5], provide unique opportunities for exploring the physics of light quarks through numerical simulations of lattice QCD (e.g., [6,7]). In this regularization, the $\Delta S = 1$ effective weak Hamiltonian renormalizes with the same pattern as in the continuum. As a consequence, in the presence of an active charm quark, the $\Delta I = 1/2$ rule in $K \to \pi\pi$ decays can be studied from the simpler $K \to \pi$ amplitudes without having to perform difficult power subtractions [8]. In addition, this exact chiral-flavor symmetry implies full $O(a)$ improvement. In this Letter, we present the results of a quenched calculation of the matrix element of the $\Delta S = 2$ effective weak Hamiltonian relevant to $K^0 - \bar{K}^0$ mixing, performed using Neuberger's implementation of GW fermions [3,4]. The bare matrix element is renormalized nonperturbatively in the regularization-independent/momentum (RI/MOM) scheme à la [9]. This calculation is the first study of the matrix element of a four-quark operator using a lattice formulation which has exact chiral-flavor symmetry at finite lattice spacing. It calls upon many of the ingredients required for studying the $\Delta I = 1/2$ rule or ϵ' (three-point functions, nonperturbative renormalization of four-quark operators, etc.), but avoids the numerically challenging computation of eye diagrams. Furthermore, because this $\Delta S = 2$ matrix element has been extensively studied with other fermion formulations (e.g., [10,11]), it provides a good test of the reliability of our approach.

In addition to using Neuberger fermions, we make a number of other improvements on the methods traditionally used to compute $\Delta S = 2$ matrix elements. In order to eliminate sizable unphysical finite-volume contributions from topology-induced fermion zero modes, we use the fact that the relevant $\Delta S = 2$ operator is purely left handed and consider correlation functions constructed from left-left quark propagators only. These propagators are free from zero-mode contributions, as the latter are chiral. Thus, to create and destroy kaons at rest, we use the time component of left-handed, quark-bilinear currents. The use of left-handed currents was proposed in [12,13] in the context of the ϵ regime of Gasser and Leutwyler [14]. We also use chiral sources in the RI/MOM nonperturbative renormalization (NPR) of our operators to avoid zero-mode contributions, which would otherwise appear. This has the added advantage of eliminating the leading chiral-symmetry-breaking contributions to the NPR Green functions.

Indirect CP violation in $K \to \pi\pi$ decays is governed by the $K^0 - \bar{K}^0$ mixing matrix element ($F_\pi = 93$ MeV),

$$\langle \bar{K}^0 | \mathcal{O}_{\Delta S=2}(\mu) | K^0 \rangle = \frac{16}{3} M_K^2 F_K^2 B_K(\mu), \tag{1}$$

with

$$\mathcal{O}_{\Delta S=2} = [\bar{s}\gamma_\mu(1-\gamma_5)d][\bar{s}\gamma_\mu(1-\gamma_5)d]. \tag{2}$$

Combined with a determination of B_K, the measurement of this CP violation provides an important hyperbolic constraint on the summit of the Cabibbo-Kobayashi-Maskawa unitarity triangle. A preliminary version of the present calculation was reported in [15]. Preliminary results obtained by the MILC collaboration with a different implementation of GW fermions were also published in the same volume [16].

Computational details.—The simulation is performed in quenched QCD with $\beta = 6.0$ and $V = 16^3 \times 32$. We use a sample of 80 gauge configurations, generated with the standard Wilson gluon action, retrieved from the repository at the "Gauge Connection" (cf. http://qcd.nersc.gov). Fermion propagators are obtained from a local source, using Neuberger's action [3,4] with bare quark masses $am = 0.040$, 0.055, 0.070, 0.085, and 0.100. The sign function of the Hermitian Wilson-Dirac operator, X, is obtained by making an optimal rational approximation [17–19], after explicit evaluation of the contributions from the lowest eigenvectors of $X^\dagger X$. The computation of the propagators thus uses nested multiconjugate gradient inversions (for more details, see [20,21]). The lattice spacing is determined from $r_0 = 0.5$ fm, where r_0 is the Sommer scale [22]. This gives $a^{-1} = 2.12$ GeV.

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Meson correlation functions and fits.—From the quark propagators, we compute the two-point function,

$$C_{JJ}(x_0) = \sum_{\mathbf{x}} \langle J_0(x) \bar{J}_0(0) \rangle, \tag{3}$$

and the three-point function,

$$C_{J\mathcal{O}J}(x_0, y_0) = \sum_{\mathbf{x},\mathbf{y}} \langle J_0(x) \mathcal{O}^{\text{bare}}_{\Delta S=2}(0) J_0(y) \rangle, \tag{4}$$

where $J_\mu = \bar{d}\gamma_\mu(1-\gamma_5)\tilde{s}$, $\tilde{q} = (1 - \frac{a}{2\rho}D)q$, and $\rho = 1.4$ (see [20,21]). $\bar{J}_0$ is obtained from J_0 with $s \leftrightarrow d$ and $\mathcal{O}^{\text{bare}}_{\Delta S=2}$ is obtained from Eq. (1) with $d \to \tilde{d}$. Statistical errors are estimated with the jackknife method.

The kaon mass M_K is determined from a fit of the time-symmetrized two-point function to the standard asymptotic form. This function is asymptotic from $ax_0 = 5$ on, and the statistically optimal fitting range for aM_K is $5 \leq ax_0 \leq 12$, 13, 14, 15, and 16 when $am = 0.040$, 0.055, 0.070, 0.085, and 0.100, respectively.

The bare bag parameter, B_K^{bare}, is obtained by fitting the ratio

$$R(x_0, y_0) = \frac{3}{8} \frac{C_{JOJ}(x_0, y_0)}{C_{JJ}(x_0) C_{JJ}(y_0)} \tag{5}$$

to the asymptotic form ($a \ll x_0 \ll y_0 \ll T$):

$$R(x_0, y_0) \to B_K^{\text{bare}}. \tag{6}$$

Asymptotic behavior sets in for $ax_0 \geq 5$ and $ay_0 \leq 27$. Because our lattice is rather short in the time direction, we have to worry about time-reversed contributions. Assuming that the time-reversed matrix element is approximately the same as the forward one and that the two-kaon energy is approximately $2M_K$ (i.e., that finite-volume effects are not significant), we find analytically that the time-reversed contributions never exceed 1.5% of the forward signal when $ax_0 \leq 6$, $ay_0 \geq 26$. We also checked this explicitly by a fit which includes time-reversed contributions and allows for finite-volume shifts on the time-reversed matrix element and the two-kaon energy. We thus use the ranges $5 \leq ax_0 \leq 6$ and $26 \leq ay_0 \leq 27$ to calculate our observables. All fits are excellent and our results for aM_K and B_K^{bare} are summarized in Table I.

Nonperturbative renormalization.—We perform all renormalizations nonperturbatively in the RI/MOM scheme à la [9]. Thus, we fix gluon configurations to Landau gauge and numerically compute appropriate, amputated forward quark Green functions with legs of momentum $p \equiv \sqrt{p^2}$. We then define the ratio

$$\mathcal{R}^{\text{RI}}(m, p^2, g_0) = \frac{\Gamma_J(m, p^2)^2}{\Gamma_{\mathcal{O}_{\Delta S=2}}(m, p^2)}, \tag{7}$$

where $\Gamma_{\mathcal{O}}$ is the amputated Green function of operator $\mathcal{O}$ obtained with left-handed quark sources and projected onto the spin-color structure of $\mathcal{O}$. We compute this ratio for all five quark masses and perform a linear chiral extrapolation in m^2 as prescribed by a next-to-leading order expansion of the Green functions in quark mass. We find that the mass dependence is very mild and is well described by this linear form. We then isolate the "perturbative part" of this ratio to get the renormalization constant appropriate for renormalizing $B_K^{\text{bare}}(a)$. A straightforward operator product expansion (OPE) in $1/p^2$ yields the following p^2 behavior for $\mathcal{R}^{\text{RI}}$:

$$\mathcal{R}^{\text{RI}}(0, p^2, g_0) = \cdots + \frac{A}{p^2} + Z^{\text{RGI}}_{B_K}(g_0) U^{\text{RI}}(p^2) + B(ap)^2 + \cdots, \tag{8}$$

where $Z^{\text{RGI}}_{B_K}(g_0)$ is the renormalization constant that takes $B_K^{\text{bare}}(a)$ to the renormalization-group invariant (RGI) parameter B_K^{RGI}, and $U^{\text{RI}}(p^2)$ describes the running in p^2 of the corresponding RI-scheme renormalization constant $Z^{\text{RI}}_{B_K}(p^2, g_0)$. The ellipses correspond to higher-order terms in the OPE and higher-order discretization errors.

To describe the running of $Z^{\text{RI}}_{B_K}(p^2, g_0)$, we use the two-loop expression obtained by combining the $\overline{\text{MS}}$ − NDR (modified minimal subtraction and naive dimensional regularization) result of [23] and the $\overline{\text{MS}}$ − NDR → RI matching result of [24], with $N_f = 0$. Thus,

$$U^{\text{RI}}(p^2) = \alpha_s(p^2)^{[\gamma_0/(2\beta_0)]} \times \left[1 + \frac{\beta_1}{\beta_0}\frac{\alpha_s(p^2)}{4\pi}\right]^{[\gamma_1^{\text{RI}}/(2\beta_1)] - [\gamma_0/(2\beta_0)]}, \tag{9}$$

with $\beta_0 = 11$, $\beta_1 = 102$, $\gamma_0 = 4$, and $\gamma_1^{\text{RI}} = 287/3 - 176\ln 2$. The strong coupling constant that we use is taken from [25] and corresponds to $\alpha_s(\mu^2_{\text{ref}}) = 0.0926(12)$ in the $\overline{\text{MS}}$ scheme with $\mu_{\text{ref}} = 94.1(3.6)/r_0$. To obtain the coupling at other scales, we integrate the two-loop running equation exactly and solve it numerically.

Our results for $\mathcal{R}^{\text{RI}}(0, p^2, g_0)$ are plotted in Fig. 1 as a function of $p^2 \equiv [(a/r_0)^2/a^2]\sum_{\mu=0}^{3}(\sin ap_\mu)^2$. Using this lattice definition of momentum significantly reduces discretization effects. Also shown is the fit of these results to the functional form given by Eqs. (8) and (9) with $Z^{\text{RGI}}_{B_K}(g_0)$, A and B as parameters, in the range of $2\ \text{GeV}^2 \leq p^2 \leq 10\ \text{GeV}^2$. This fit yields our central value for $Z^{\text{RGI}}_{B_K}(g_0)$. We find that the OPE and discretization error terms kept in Eq. (8) are sufficient to describe the data in this range. The fit actually describes the data in a

TABLE I. Meson masses and $B_K^{\text{bare}}(a)$ vs quark mass.

am	aM_K	$B_K^{\text{bare}}(a)$
0.040	0.253(5)	0.70(7)
0.055	0.288(4)	0.71(6)
0.070	0.321(3)	0.73(5)
0.085	0.352(3)	0.74(4)
0.100	0.382(2)	0.75(4)

much larger range, indicating that the retained terms dominate. We also perform a fit to the extended momentum range $0.65\ \text{GeV}^2 \leq p^2 \leq 15.9\ \text{GeV}^2$ with additional $1/p^4$ and $(ap)^4$ terms, and a fit using the continuum p^2 in the range $2\ \text{GeV}^2 \leq p^2 \leq 10\ \text{GeV}^2$. All fits are excellent and produce compatible results. The renormalization constants at $4\ \text{GeV}^2$ in the RI and $\overline{\text{MS}} - \text{NDR}$ schemes are obtained by multiplying $Z_{B_K}^{\text{RGI}}(g_0)$ by the appropriate two-loop running expressions $U^{\text{RI}}(4\ \text{GeV}^2)$ and $U^{\text{NDR}}(4\ \text{GeV}^2)$, respectively, with N_f and α_s chosen as above. $U^{\text{NDR}}(p^2)$ is given by Eq. (9) with $\gamma_1^{\text{RI}} \rightarrow \gamma_1^{\text{NDR}} = -7$ for $N_f = 0$.

Physical results.—The central values for $B_K^{\text{bare}}(a)$, $Z_{B_K}^{\text{RGI}}(g_0)$, $Z_{B_K}^{\text{NDR}}(4\ \text{GeV}^2, g_0)$ and $Z_{B_K}^{\text{RI}}(4\ \text{GeV}^2, g_0)$ are obtained as described above. While statistical and systematic errors in the B parameters and renormalization constants will be correlated in our final results, we also wish to give results for the renormalization constants themselves. The systematic errors on these constants take into account the following sources of uncertainty: the errors on $\alpha_s(\mu_{\text{ref}}^2)$ and μ_{ref}, given above; the variation due to a $\pm 10\%$ uncertainty on the lattice spacing, which is typical in quenched calculations [26]; the difference between the renormalization constants obtained using the lattice and continuum definitions of quark momentum. The latter yields an estimate of discretization errors which turns out to be the dominant uncertainty. We take it to be a symmetric error. Adding all of these variations in quadrature, we obtain ($\rho = 1.4$)

$$\begin{aligned} Z_{B_K}^{\text{RGI}}(g_0) &= 1.261(9)^{+11}_{-10}, \\ Z_{B_K}^{\text{NDR}}(4\ \text{GeV}^2, g_0) &= 0.908(6)^{+7}_{-6}, \\ Z_{B_K}^{\text{RI}}(4\ \text{GeV}^2, g_0) &= 0.897(6)^{+7}_{-6}. \end{aligned} \tag{10}$$

We now turn to B_K. Our lightest pseudoscalar meson is very close to having the mass of the kaon. Since our results for B_K are linear in M_K^2, we extrapolate them linearly to the physical point, as shown in Fig. 2. To extrapolate to lighter quarks, chiral logarithms should be taken into account. Thus, we also perform a fit to the functional dependence predicted by one-loop quenched chiral perturbation theory (QχPT) [27], supplemented by an $(M/4\pi F)^4$ term to parametrize higher-order contributions, which are expected for our pseudoscalar meson masses. Here, F is the leading-order leptonic decay constant, which we take to be F_π, and M is the leading-order pseudoscalar meson mass, which we set equal to meson masses obtained in our simulation. The fit parameters are the RGI value of the chiral-limit B parameter, B^{RGI}, the scale of the one-loop logarithm and the coefficient of the $(M/4\pi F)^4$ term. The fit, also shown in Fig. 2, is only meant to indicate how a chiral extrapolation could lead to much smaller values of the B parameter in the chiral limit, such as the ones found in the NLO, large-N_c calculations of [28], based on the minimal hadronic approximation, and of [29], based on the extended Nambu-Jona-Lasinio model. We would need results at lower masses and with better statistics to confirm the presence of the logarithm and enable a reliable extrapolation to the chiral limit. For the time being, we quote, in the quenched approximation, $B^{\text{RGI}} = 0.53(11)^{+4+30}_{-3-0}$, where the central value is obtained from the χPT fit. The first error here is statistical and is obtained by performing a jackknife analysis on the chiral extrapolation of the product $Z_{B_K}^{\text{RGI}}(g_0)B_K^{\text{RGI}}$ computed at our five quark masses; the second results from first implementing, on this same product, the variations that were discussed above Eq. (10), then propagating these variations to B^{RGI} through the chiral extrapolation and finally adding the resulting differences from our central value in quadrature; the third is the difference between the central values of linear and χPT fits.

As evident from Fig. 2, the value of B_K^{RGI} at the physical point is insensitive to the choice of functional form in the

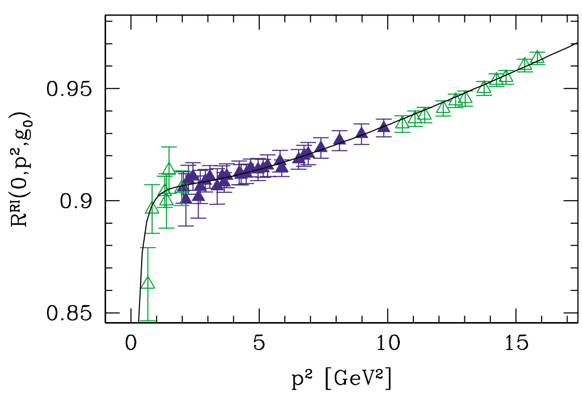

FIG. 1 (color online). $\mathcal{R}^{\text{RI}}(0, p^2, g_0)$ vs p^2. The solid triangles are the points used in the fit to Eq. (8). The curve is the result of the fit.

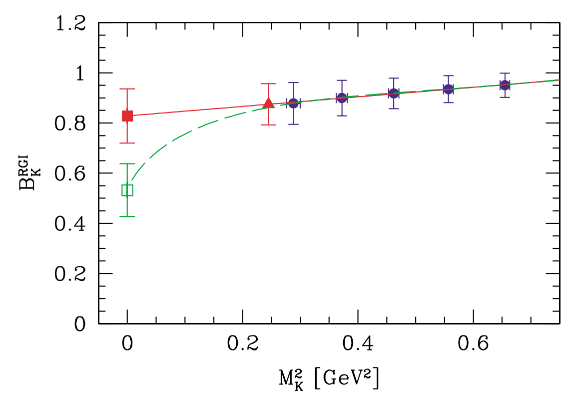

FIG. 2 (color online). B_K^{RGI} vs M_K^2. The solid circles are the results of our simulation. They are fitted to a line (solid curve) and a QχPT expression (dashed curve). The solid triangle is the value of B_K^{RGI} at the physical point. The squares correspond to the chirally extrapolated values B_K^{RGI} as obtained from the linear (solid square) and QχPT (open square) fits.

fit. We thus use the simpler linear fit to determine $B_K^{\rm RGI}$ at that point. We allow for a 20% uncertainty in the determination of the strange quark mass, which is typical of the variations observed in quenched calculations of this quantity [26]. In addition, we account for statistical errors and the systematic variations that were discussed above Eq. (10) in the same way as for $B^{\rm RGI}$. We do not include a discretization error on the bare value of B_K, because it cannot be estimated from results obtained at a single value of the lattice spacing. However, B parameters are ratios of very similar matrix elements, in which some discretization errors should cancel. Moreover, the limited experience that we have with Neuberger fermions suggests that discretization errors on quantities such as F_K are small at $\beta = 6.0$ [30].

Our results for B_K in various schemes are

$$\begin{aligned} B_K^{\rm RGI} &= 0.87(8)^{+2+14}_{-1-14}, \\ B_K^{\rm NDR}(4\ {\rm GeV}^2) &= 0.63(6)^{+1+10}_{-1-10}, \\ B_K^{\rm RI}(4\ {\rm GeV}^2) &= 0.62(6)^{+1+10}_{-1-10}, \end{aligned} \tag{11}$$

where the first error is statistical and the second corresponds to the systematic uncertainties discussed in the preceding paragraph. The third is obtained by combining in quadrature two errors common to all quenched calculations. The first of these is associated with the fact that our kaon is composed of degenerate quarks with masses $\sim m_s/2$, instead of an s and a d quark. It is thought to be roughly 5% on the basis of χPT estimates [31]. The second is due to quenching. We have accounted for part of this error by varying the lattice spacing and the strange quark mass. Though we find that the result of these variations is very much smaller than the 15% quenching error suggested in [31], we retain Sharpe's conservative estimate.

The results of Eq. (11) are in excellent agreement with world averages for this quantity (e.g., [10,11]), which are based on quenched, staggered results [32–34]. This comparison is performed after omitting the second systematic error, which is common to all calculations. However, even without this error, our precision is not sufficient to exclude the rather low values found with quenched domain-wall fermions [35,36].

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