Minimal Stochastic Model for Fermi's Acceleration

Freddy Bouchet
Dipartimento di Fisica, Università "La Sapienza," Piazzale Aldo Moro 2, I-00185 Roma, Italy

Fabio Cecconi and Angelo Vulpiani
Dipartimento di Fisica, Università "La Sapienza" & INFM UdR Roma-1 and Center for Statistical Mechanics and Complexity, Piazzale Aldo Moro 2, I-00185 Roma, Italy
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We introduce a simple stochastic system able to generate anomalous diffusion for both position and velocity. The model represents a viable description of the Fermi's acceleration mechanism and it is amenable to analytical treatment through a linear Boltzmann equation. The asymptotic probability distribution functions for velocity and position are explicitly derived. The diffusion process is highly non-Gaussian and the time growth of moments is characterized by only two exponents ν_x and ν_v. The diffusion process is anomalous (non-Gaussian) but with a defined scaling property, i.e., $P(|\mathbf{r}|, t) = 1/t^{\nu_x} F_x(|\mathbf{r}|/t^{\nu_x})$ and similarly for velocity.

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About half a century ago, Fermi proposed an acceleration mechanism for interstellar particles, now referred to as Fermi's acceleration [1], to explain the very high energy of cosmic rays. Nowadays, Fermi's acceleration remains one of the relevant explanations for several phenomena in plasma physics [2] and astrophysics [3,4]. In Fermi's mechanism, at variance with diffusion in real space (e.g., Lorentz's gas [5,6]), it is the velocity that undergoes diffusion due to the presence of a stochastic acceleration. This original idea [1] found also a number of applications in the theory of dynamical systems, because it offers a simple but not trivial way to generate chaotic systems [7–13] to describe the dynamics of comets [14] and the motion of charged particles into electromagnetic fields [15].

This Letter studies Fermi's mechanism through a simple model, in which a particle may absorb kinetic energy (accelerate) through collisions against moving scatterers. The purpose of the model is to provide a description of the acceleration mechanism without specific assumptions on the interstellar media (e.g., the presence of turbulent magnetic fields or matter fractal distribution). Since we are interested mainly in the general features of the diffusion process in phase space, we consider only nonrelativistic classical particles, neglecting the details of their interactions with magnetic fields. Our model is able to produce an acceleration process characterized by anomalous diffusion both in velocity and position: $\langle|\mathbf{v}(t)|^2\rangle \sim t^{2\nu_v}$, $\langle|\mathbf{x}(t)|^2\rangle \sim t^{2\nu_x}$ ($\nu_v > 1/2$, $\nu_x > 1/2$) and non-Gaussian distributions. For simplicity we restrict to a two dimensional system [$\mathbf{x}(t) \in \mathbf{R}^2$] and circular obstacles with radius a. These obstacles represent the regions where the magnetic field irregularities, responsible for the scattering, are localized. We shall assume that they are randomly distributed in the plane with uniform density ρ. Their velocity $\mathbf{V} = (V\cos\phi, V\sin\phi)$ is chosen according to an isotropic probability $G(V)dVd\phi/2\pi$, with the only requirement that $\langle V^2\rangle < \infty$. We furthermore assume that the influence of the particles on the magnetic scatterers is so negligible that the velocity of the latter is kept unchanged during collisions. In the reference frame of the scatterers, where the energy of the particle remains constant, the collision is completely determined by the trajectory deflection angle and collision time. Such a deflection angle $\alpha = \alpha(\mathbf{V}, \mathbf{v}, b, H)$ depends on particle and obstacle velocity, on the impact parameter b, and on the shape of the magnetic field H inside the obstacle.

After a collision, the new particle velocity is $\mathbf{v}_S^A = \mathbf{M}(\alpha)\mathbf{v}_S^B$, where the superscripts A and B stand for "after" and "before" the collision, respectively, while subscript S refers to velocities in the scatterers reference frame and the rotation matrix $\mathbf{M}(\alpha)$ describes the scattering deflection. In the fixed reference frame, $\mathbf{v} = \mathbf{v}_S + \mathbf{V}$, we have

$$\mathbf{v}^A = \mathbf{V} + \mathbf{M}(\alpha)(\mathbf{v}^B - \mathbf{V}).$$

A further simplification occurs when the density of the scatterers ρ is small. Indeed, in the low density limit ($\rho \sim l_0^{-2} \ll a^{-2}$), the mean free path is much larger than the average distance between nearest obstacles l_0, and two successive collisions can be safely assumed to be independent. Such a simplification allows describing the evolution of the system as a stochastic map:

$$\begin{aligned} \mathbf{r}_{n+1} &= \mathbf{r}_n + \mathbf{v}_n \Delta t_n, \qquad t_{n+1} = t_n + \Delta t_n, \\ \mathbf{v}_{n+1} &= \mathbf{V} + \mathbf{M}(\alpha)(\mathbf{v}_n - \mathbf{V}), \end{aligned} \tag{1}$$

where $\Delta t_n = \ell_n/|\mathbf{v}_n|$ and ℓ_n are the time and the free path between two consecutive collisions, respectively. Under the hypothesis of independent collisions, well verified at low densities, α and Δt_n become independent random variables, whose distribution will be determined by the distribution of the obstacle velocities $\mathbf{V}$ and impact parameters b. For the sake of simplicity, we suppose that the probability distribution $Q(\alpha)$ for α is independent of $\mathbf{V}$ and is an even function (for symmetry reasons).

Let us now specify, the probability distribution of Δt. If the scatterers were at rest, a particle with speed v would encounter uniformly distributed obstacles, and then the variable Δt would follow an exponential distribution law: $P(\Delta t) = \lambda \exp(-\lambda \Delta t)$ with a decay rate $\lambda = 2a\rho \mathbf{v}$. However, in the case of moving obstacles, a particle will encounter preferentially those obstacles with a velocity direction opposite to its own velocity. Then, the rate λ has to be replaced by $\lambda(\mathbf{v}, \mathbf{V}) = 2a\rho|\mathbf{v} - \mathbf{V}|$. Therefore the probability that a particle with velocity $\mathbf{v}$ undergoes a collision with an obstacle of velocity $\mathbf{V}$, after a time Δt from the previous collision is

$$P_{\mathbf{V}}(\Delta t \mid \mathbf{v}) = 2a\rho|\mathbf{v} - \mathbf{V}|G(V)\exp\{-\bar{\lambda}(\mathbf{v})\Delta t\}, \quad (2)$$

with $\bar{\lambda}(\mathbf{v}) = 2a\rho\langle|\mathbf{v} - \mathbf{V}|\rangle_{\mathbf{V}}$, where $\langle\cdots\rangle_{\mathbf{V}}$ indicates the average on $\mathbf{V}$ [16]. Of course more general laws can be considered to take into account the presence of a minimal collision time, but our simulations show that the results are fairly independent of the details of $P(\Delta t)$.

The above system might look somehow trivial when considering its statistical properties as functions of the number of collisions. Noting that $\mathbf{v}_{n+1}$ is obtained from $\mathbf{v}_n$ by a random rotation plus a random shift, one can expect, from the central limit theorem, that $\langle|\mathbf{v}_n|^2\rangle \sim n$ and analogously $\langle|\mathbf{r}_n|^2\rangle \sim n$; of course the probability distribution functions (PDFs) of $\mathbf{v}_n$ and n are Gaussian. However, the proper question is about the dependence in time rather than in n, so the particles at the same time t experience a different number of collisions, and therefore the PDFs of $\mathbf{v}(t)$ and $\mathbf{r}(t)$ are not expected to be Gaussian. The time after n collisions, $t = \sum_{k=1}^{n} \Delta t_k$, is the sum of random quantities and its dependence on n can be worked out via the following scaling argument [17]. Since $\Delta t_k \sim \ell/\sqrt{\langle|\mathbf{v}_k|^2\rangle} \sim 1/\sqrt{k}$, then $t \sim \sum_{k=1}^{n} k^{-1/2} \sim \sqrt{n}$; $\langle|\mathbf{r}_n|^2\rangle \sim n$ corresponds to the scaling laws:

$$\langle|\mathbf{v}(t)|^2\rangle \sim t^2 \quad \text{and} \quad \langle|\mathbf{r}(t)|^2\rangle \sim t^2. \quad (3)$$

In the following, we will show that this "mean field" scaling behavior is in fact correct.

Let us now present a simple but relevant remark: an easy computation gives

$$\langle|x(t) - x(0)|^2\rangle \simeq 2\int_0^t \int_0^{t-t'} \langle|v_x(t')|^2\rangle R_x(t', \tau)dt'd\tau, \quad (4)$$

with $R_x(t, \tau) = \langle v_x(t)v_x(t + \tau)/\langle|v_x(t)|^2\rangle$. In Lorentz gas with fixed obstacles and elastic collisions, $\langle|v_x(t')|^2\rangle$ remains bounded; thus the integration over t' yields a proportionality to t. In addition stationarity makes $R_i(t', \tau)$ independent of t' and an anomalous diffusion can originate only by a long time tail of $R_i(\tau)$ [i.e., slower than $O(\tau^{-1})$].

By contrast, in systems with Fermi's acceleration which are nonstationary, $R_x(t, \tau)$ depends also on t, and the scaling laws, $\langle|\mathbf{v}(t)|^2\rangle \sim t^{2\nu_v}$ and $\langle|\mathbf{r}(t)|^2\rangle \sim t^{2\nu_x}$, could not be trivially related [see (4)]. In this case, one can derive only the bound:

$$\nu_x \leq \nu_v + 1,$$

the equality holding in the presence of a very strong time correlation among the velocities.

A Boltzmann-like description of our model can be carried out under the hypothesis of the independence of consecutive collisions, already employed to write the dynamics (1) and (2). The spatial homogeneous probability $f(\mathbf{v}, t)$, that a particle has a velocity $\mathbf{v}$ at time t, evolves according to the equation

$$\partial_t f(\mathbf{v}, t) = -\bar{\lambda}(\mathbf{v})f(\mathbf{v}, t) + \int d\mathbf{u}T(\mathbf{v}, \mathbf{u})f(\mathbf{u}, t), \quad (5)$$

where, from Eqs. (1) and (2), the transition probability T reads

$$T(\mathbf{v}, \mathbf{u}) = \langle\lambda(\mathbf{u}, \mathbf{V})\delta[\mathbf{v} - \mathbf{V} - \mathbf{M}(\alpha)(\mathbf{u} - \mathbf{V})]\rangle_{\mathbf{V},\alpha}. \quad (6)$$

Equation (5), at variance with the usual Boltzmann equation, is linear because of the independence between obstacles and particles. Moreover, the isotropy of the scatterer distribution insures that T is only a function of the moduli v, v^B and of the angle $\theta - \theta^B$ between $\mathbf{v}$ and $\mathbf{v}^B$: $T = T(v, v^B, \theta - \theta^B)$. Therefore it is convenient to expand f in a Fourier series of the angular variable: $f(v, \theta) = \sum_k f_k(v)\exp(ik\theta)$. The linearity of Eq. (5) decouples Fourier modes f_k and we obtain $\partial_t f_k(v, t) = -\lambda(v)f_k + \int dv^B T_k(v, v^B)f_k(v^B, t)$ with $T_k(v, v^B) = \int d\theta \exp(ik\theta)T(v, v^B, \theta)$.

We can restrict to the asymptotic time behavior because the particles accelerate and their distribution is rapidly dominated by velocities $v \gg V$. Then an asymptotic expansion of the operator T is possible in the small parameter V/v.

We first consider the evolution of an isotropic density $f(v, \theta) = f(v)$ by introducing the PDF of the velocity modulus, $v = |\mathbf{v}|$, $g(v, t) = 2\pi v f(|\mathbf{v}|)$. We can then substitute $\delta(\mathbf{v} - \mathbf{V} - \mathbf{M}(\alpha)(\mathbf{v}^B - \mathbf{V}))$ with $\delta[\mathbf{v}^B - \mathbf{v} - (1 - M(\alpha))\mathbf{V}]$ in Eq. (6) and, at second order in v/V, we obtain [18]

$$\partial_t g(v, t) = Dv\partial_v^2 g, \quad (7)$$

where $D = a\rho\sigma_F\langle V^2\rangle_{\mathbf{V}}$ and $\sigma_F = \langle 1 - \cos\alpha\rangle_\alpha$. The asymptotic solution of this Fokker-Planck equation may be explicitly derived [18] and it converges toward the scaling function $g_a(v, t) = h[v/(Dt)]/(Dt)$. A direct substitution into Eq. (7) shows that the scaling function h verifies the equation: $\xi h''(\xi) + \xi h'(\xi) + h(\xi) = 0$, whose solution is $h(\xi) = \xi \exp(-\xi)$. Thus $g_a(v, t) = v/(Dt)^2 \exp\{-v/(Dt)\}$ and the moments of v are given by $\langle v^n\rangle \sim (n + 1)!(Dt)^n$.

We now consider the evolution of a nonisotropic Fourier mode $f(v, \theta) = \exp(ik\theta)f_k(v)$. We can then substitute $\delta[\mathbf{v} - \mathbf{M}(\alpha)\mathbf{v}^B - (1 - \mathbf{M}(\alpha))\mathbf{V}]$ with $\exp(-ik\alpha)\times \delta[\mathbf{v}^B - \mathbf{v} - (1 - M(\alpha))\mathbf{V}]$, in Eq. (6) and performing the expansion at small v/V we obtain

$$\partial_t f_k(v) = -L_k f_k(v) \quad (8)$$

at leading order in V/v, with $L_k = 2a\rho\sigma_{F,k}$ and $\sigma_{F,k} = \langle 1 - \cos(k\alpha)\rangle_\alpha$ (we have exploited even symmetry of the α distribution Q). Thus for any initial distribution function, f_k relaxes exponentially. This is physically clear, as few collisions randomize the phase of the velocity. The proportionality to v of the relaxation rate is a consequence of the fact that the number of collisions per unit time is proportional to v. We note that the neglected higher order terms in V/v, in Eq. (8), are a transport and a diffusion term, not relevant for large v.

This result shows that any distribution rapidly becomes isotropic; it also allows one to compute the velocity autocorrelation function and some properties of the spatial diffusion. For large t, the velocity autocorrelation can be expressed in terms of the asymptotic distribution g_a and of the solution of the Boltzmann equation (5), with a deltalike initial condition $\delta(\mathbf{v}' - \mathbf{v})$. Using Eq. (8), we can compute this solution. Using this result, we obtain the expression for the autocorrelation function:

$$\langle v_x(t)v_x(t+\tau)\rangle = 3(Dt)^2/(1 + DL_1 t\tau)^4. \quad (9)$$

The diffusion properties of velocity can be derived directly by Eq. (9) computed at $t = 0$ and from the symmetry $x \leftrightarrow y$

$$\langle|\mathbf{v}(t)|^2\rangle = 6(Dt)^2. \quad (10)$$

The algebraic decay of the function (9) is fast enough to make the integral (4) convergent, then

$$\langle|x(t) - x(0)|^2\rangle \sim \frac{2D}{L_1}t^2. \quad (11)$$

The above results (9)–(11) agree with the simple argument leading to Eq. (3). In order to study the statistical properties of the particle position, we consider the evolution of the PDF, $f(\mathbf{r}, \mathbf{v}, t)$, for the velocity and position. The generalization of Eq. (5) is the inhomogeneous Boltzmann equation where $\partial_t f$ is replaced by $\partial_t f(\mathbf{r}, \mathbf{v}, t) + \mathbf{v} \cdot \nabla_{\mathbf{r}} f$. We introduce the distribution $g(r, \phi, v, \theta, t) = 4\pi^2 r v f(\mathbf{r}, \mathbf{v}, t)$ in polar coordinates, $\mathbf{r} = (r\cos\phi, r\cos\phi)$, $\mathbf{v} = (v\cos\theta, v\cos\theta)$. We limit the discussion to the isotropic case; that is $g(r, \phi, v, \theta) = g(r, v, \theta - \phi)$. Again, it is convenient to develop the angular dependence in Fourier modes: $g(r, \phi, v, \theta) = \sum_k g_k(r, v)\exp(ik(\theta - \phi))$.

The evolution equation for each mode g_k can be written down and studied in the long time limit, but we do not report here the detailed derivation. We can prove that g_0 is dominant for large t and converges toward a scaling function $h_0[L_1^{1/2} r/(D^{1/2}t), v/(Dt)]L_1^{1/2}/(D^{3/2}t^2)$ [19] which verifies the equation

$$2h_0 + \eta\partial_\eta h_0 + \xi\partial_\xi h_0 + \xi\partial_\xi^2 h_0 + \frac{\xi}{4}\left[\partial_\eta^2 h_0 - \partial_\eta\left(\frac{h_0}{\eta}\right)\right] = 0. \quad (12)$$

We were not able to explicitly solve Eq. (12); however we computed all the moments of h_0 by recurrence. Indeed, if we define $a_{k,n} = \int_0^\infty dr dv v^k r^n h_0(r, v)$, we obtain $(n + k)a_{k,n} = k(k+1)a_{k-1,n} + n^2/4 a_{k+1,n-2}$. Using that $\int_0^\infty d\eta h_0(\eta, \xi) = h(\xi) = \xi\exp(-\xi)$, we have $a_{k,0} = (k+1)!$. The recurrence relation then, for instance, gives $\langle r^2\rangle = 2\langle|x(t) - x(0)|^2\rangle \sim Dt^2/L_1$, $\langle r^4\rangle \sim 8D^2t^4/(3L_1^2)$, and so on.

Numerical simulations confirm our theoretical results obtained via the Boltzmann equation for the scaling of the moments of position and velocity. Simulations were run for a set of $N = 10^6$ particles starting from random initial positions and velocities well localized in a region of the phase space with size small compared with l_0^2 and V_0^2. For a particle with velocity $\mathbf{v}$, we randomly select a scatter velocity $\mathbf{V}$ from $G(V) = \delta(V - V_0)/2\pi$ and a scattering angle α uniformly distributed in $[0, 2\pi]$. Then the collision time Δt_n was drawn from the law (2) and finally particle position and velocity were updated via the rule (1). Figures 1 and 2 show, for different times, the rescaled PDF for the velocity modulus $v = |\mathbf{v}|$ and x coordinate. The collapse of the curves is very good and the exponential tails agree with our theoretical predictions. The qualitative scenario corresponding to $\nu_x = \nu_v = 1$ and exponential tails remains unaffected by using the generic form of $P_{\mathbf{V}}(\Delta t, \mathbf{v})$, provided it keeps the exponential decay for large Δt.

We stress that the diffusion process is anomalous, since ν_x is different from $1/2$ (we discuss only the diffusion for the position but similar considerations hold for velocity). The existence of a unique exponent characterizing the growth of the different moments $\langle|\mathbf{r}(t)|^n\rangle \sim t^{n\nu_x}$ is somehow peculiar and in Ref. [20] is called weak anomalous diffusion, because the most general anomalous diffusion (strong anomalous diffusion) implies that $\langle|\mathbf{r}(t)|^n\rangle \sim t^{n\nu_x(n)}$ with $\nu_x(n)$ a nonconstant parameter. Therefore all the PDFs cannot be rescaled onto a single curve.

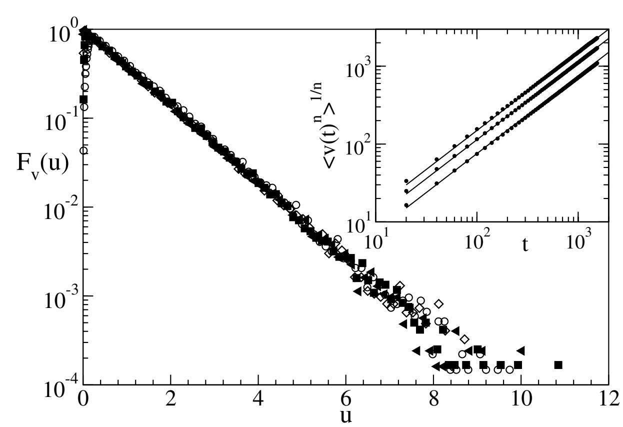

FIG. 1. PDF of rescaled particle velocity $u = |\mathbf{v}|/\sqrt{\langle v(t)^2\rangle}$ at times $t = 80$, 320, 1280, and 5120, corresponding to circles, squares, diamonds, and triangles, respectively. The perfect collapse is in accordance with the theoretical results. Inset: $\langle v(t)^n\rangle^{1/n}$ vs t, for $n = 2$, 4, and 6 from bottom to top; straight lines have a slope of 1.

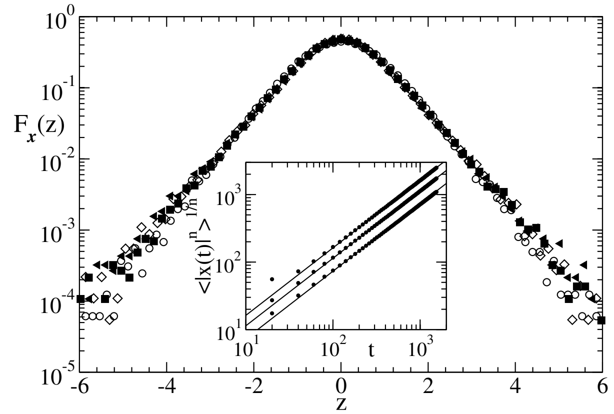

FIG. 2. PDF of the rescaled coordinate $z = x/\sqrt{\langle x(t)^2 \rangle}$ at times $t = 80$, 320, 1280, and 5120 with the same symbols as Fig. 1. Inset: $\langle x(t)^n \rangle^{1/n}$ vs t for $n = 2$, 4, and 6 from bottom to top; straight lines have a slope of 1.

The model can be made more realistic taking into account possible energy losses due to interactions with the medium and energy irradiation. This dissipation is mimicked by rescaling the velocity components by a factor $\sqrt{1-\gamma}$ after each collision, where $\gamma = |\mathbf{v}^A - \mathbf{v}^B|/[\tau_c(v^A + v^B)]$ with τ_c a typical time of the collision. The presence of dissipation introduces neither substantial modification on the tailed structure of position and velocity PDFs, nor changes their scaling behavior. Even different forms of γ do not affect the global scenario.

In summary, we introduced a simplified treatable model of Fermi's acceleration. It is remarkable that the system presents an anomalous (super) diffusion for both position and velocity, which is robust under changes of the details, and it is characterized by only two exponents $\nu_x = \nu_v = 1$ and by the PDFs' scaling behavior $P(|\mathbf{r}|, t) = 1/t^{\nu_x} F_x(|\mathbf{r}|/t^{\nu_x})$, $P(|\mathbf{v}|, t) = 1/t^{\nu_v} F_v(|\mathbf{v}|/t^{\nu_v})$. Furthermore, as a consequence of nontrivial correlations, the two exponents ν_x and ν_v are not related by a simple dimensional argument.

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[16] Consider the case with two kinds of obstacles with respective decay rates λ_1 and λ_2. Let $P_1(t)$ and $P_2(t)$ the probability to have an encounter between times t and $t + \Delta t$ for events of types 1 or 2, respectively. Let $N_i(t) = 1 - \int_0^t P_i(u) du$ for $i = 1, 2$, the probability not to have had an encounter of type i before t. Then $dN_i/dt = -\lambda_i(N_1 + N_2)$. Thus $d(N_1 + N_2)/dt = -(\lambda_1 + \lambda_2) \times (N_1 + N_2)$. $N_1 + N_2$ is exponential with decay rate $\lambda_1 + \lambda_2$ and $P_i(t) = \lambda_i \exp(-(\lambda_1 + \lambda_2)t)$. The generalization of this result to an infinite number of rates gives the result (2).
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[19] Using the scaling solution, we have $g_k = h_k(\eta, \xi)/t^k$, $\partial_t g_k = A(\eta, \xi)/t^{k+1}$, and $L_k v g_k = A(\eta, \xi)/t^{k-1}$. Thus the neglected terms are of order $1/t^2$ times the conserved ones; this is a good check of our approximations.
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