Anomalous Fluctuations in Phases with a Broken Continuous Symmetry

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It is shown that the Goldstone modes associated with a broken continuous symmetry lead to anomalously large fluctuations of the zero field order parameter at any temperature below T_c. In dimensions $2 < d < 4$, the variance of the extensive spontaneous magnetization scales as L^4 with the system size L, independent of the order parameter dynamics. The anomalous scaling is a consequence of the $1/q^{4-d}$ divergence of the longitudinal susceptibility. For ground states in two dimensions with Goldstone modes vanishing linearly with momentum, the dynamical susceptibility contains a singular contribution $(q^2 - \omega^2/c^2)^{-1/2}$. The dynamic structure factor thus exhibits a critical continuum above the undamped spin wave pole, which may be detected by neutron scattering in the Néel phase of 2D quantum antiferromagnets.

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It is one of the basic properties of any thermodynamic system that the fluctuations $\mathrm{Var}\hat{A} = \langle \hat{A}^2 \rangle - \langle \hat{A} \rangle^2 \sim V$ of an extensive variable $\hat{A}$ scale linearly with the system volume V. This property guarantees that the intensive variables $\hat{A}/V$ are self-averaging, with rms fluctuations vanishing proportional to $V^{-1/2}$. Physically, the extensive nature of $\mathrm{Var}\hat{A} \sim V$ is due to the existence of a finite, microscopic correlation length ξ. A large system can thus be partitioned into an extensive number of V/ξ^d subvolumes which are statistically independent. Since the variance in each subvolume is finite, the central limit theorem then quite generally implies a Gaussian distribution for $\hat{A}$ with a variance of order V, in agreement with the standard Einstein theory of fluctuations in macroscopic thermodynamics.

The above argument indicates that the linear scaling $\mathrm{Var}\hat{A} \sim V$ may break down only at a critical point of a continuous phase transition where the correlation length ξ diverges. In fact, from the standard relation $\mathrm{Var}\hat{M} = V \cdot T\chi$ between the fluctuations of the total "magnetization" $\hat{M}$ and the corresponding linear susceptibility, the finite size scaling $\chi(T_c, L) \sim L^{2-\eta}$ of the susceptibility right at T_c [1] implies a nontrivial dependence $\mathrm{Var}\hat{M} \sim L^{d+2-\eta}$ on system size L at the critical point. Below, it will be shown that anomalous fluctuations of the order parameter are present not only *at* T_c but in fact at *any* temperature below T_c provided the broken symmetry is continuous. This is a result of the presence of Goldstone modes, which imply that correlations decay algebraically below T_c with exponents that are independent of temperature. A specific example of recent interest are fluctuations of the condensate number $\hat{N}_0$ in a Bose-Einstein condensate. Using a weak coupling Bogoliubov approach, Giorgini *et al.* have shown [2] that $\mathrm{Var}\hat{N}_0 \sim T^2 L^4$ scale anomalously at low temperatures. In fact, this result also applies to strongly interacting superfluids, being essentially a consequence of the long wavelength phase fluctuations [3]. In the present Letter, we will show that the anomalous fluctuations in a Bose-Einstein condensate are just one particular example of a rather general phenomenon which appears in any phase with a broken continuous symmetry. Consider, for example, an isotropic ferromagnet in zero field below T_c. Because of the invariance under spin rotations, it costs no energy to rotate the direction of the magnetization vector. Provided that the microscopic interactions are short ranged, this implies that the transverse susceptibility $\chi_\perp(q)$ diverges precisely as $1/q^2$ at small wave vectors $q \to 0$. As a result, there are strong fluctuations of the *direction* of the magnetization, leading to a complete destruction of long range order at finite temperature in dimensions $d \le 2$, the well-known Mermin-Wagner-Hohenberg theorem. Regarding the *magnitude* of the magnetization, the naive expectation is that its fluctuations are just like that of a standard thermodynamic variable because there is a finite restoring force for deviations from the equilibrium value. However, as was noted a long time ago by Patashinski and Pokrovski [4], the inevitable coupling between longitudinal and transverse order parameter fluctuations entails that the longitudinal susceptibility is also singular at small wave vectors, diverging as $\chi_\parallel(q) \sim 1/q^{4-d}$ in dimensions $2 < d < 4$. Our aim in the following is to show that (i) the $1/q^{4-d}$ divergence of the longitudinal susceptibility leads to anomalous fluctuations $\mathrm{Var}\hat{M}_s \sim L^4$ of the zero field order parameter in $2 < d < 4$ and $T \neq 0$ for an arbitrary broken continuous symmetry phase. For superfluids, the corresponding *relative* condensate fluctuations are universal at low T. (ii) There is an analog of the singular nature of $\chi_\parallel$ for zero temperature phases with a broken continuous symmetry. In two dimensions and with Goldstone modes whose frequencies $\omega = cq$ vanish linearly with momentum $q = |\vec{q}|$, the dynamical susceptibility has a singular contribution $(q^2 - \omega^2/c^2)^{-1/2}$ as noted first by Sachdev on the basis of a $1/N$ expansion [5]. The dynamic structure factor of 2D quantum antiferromagnets thus exhibits a critical continuum above

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the standard δ-function spin wave peak. This effect may be observed in high resolution neutron scattering experiments.

As an effective description of an arbitrary phase with a broken continuous symmetry at finite temperature, we use the nonlinear σ model [6]. It describes directional fluctuations of an order parameter $\vec{\psi}(x) = m_s^{(0)}\vec{\Omega}(x)$ with a fixed magnitude $m_s^{(0)}$ in terms of an N-component unit vector $\vec{\Omega}(x)$. At zero external field, the effective action for the fluctuations of $\vec{\Omega}$ is

$$S[\Omega] = \frac{\rho_s}{2T}\int d^d x [\nabla\vec{\Omega}(x)]^2 \qquad |\vec{\Omega}(x)|^2 = 1, \tag{1}$$

with the spin stiffness (or helicity modulus) ρ_s as the single phenomenological parameter. In a finite system of volume $V = L^d$, there is of course no spontaneous magnetization at zero field. The breaking of a continuous symmetry below T_c, however, shows up in the integrated $O(N)$-symmetric correlation function at finite volume

$$\int_V d^d x \langle\vec{\psi}(x)\cdot\vec{\psi}(0)\rangle_L = V m_L^2 \to V m_s^2, \tag{2}$$

defining an intensive nonzero order parameter m_L^2 which approaches the spontaneous magnetization m_s^2 of the infinite system in the thermodynamic limit $L \to \infty$. For superfluids, this is just the number of particles in the condensate. In dimensions $d > 2$, where m_s is nonzero at finite T, the leading long distance behavior of the two point function $G(r) = \langle\vec{\psi}(x)\cdot\vec{\psi}(0)\rangle$ may be obtained from a simple Gaussian spin wave calculation. Following a standard procedure [6], the unit vector $\vec{\Omega}(x) = (\vec{\Pi}(x), \sqrt{1-\vec{\Pi}(x)^2})$ is decomposed into $N-1$ "transverse" Goldstone fields $\Pi_a(x)$, $a = 1, \ldots N-1$ and a longitudinal component $\Omega_N(x) = \sqrt{1-\vec{\Pi}(x)^2}$. At low enough temperatures, where spin wave interactions can be neglected, the Π fields are Gaussian random variables with variance $\langle\Pi_a(q)\Pi_b(q')\rangle = \delta_{a,b}\delta_{q,-q'}T/\rho_s q^2$. The zero field correlation function below T_c,

$$G(r) = m_s^2[1 + C_\parallel(r) + (N-1)\cdot C_\perp(r)], \tag{3}$$

is thus split into longitudinal and transverse parts, with $m_s^2 = G(\infty)$ the renormalized value of the spontaneous magnetization. To lowest nontrivial order in the small fluctuations Π, the transverse correlation function decays proportional to $T/\rho_s r^{d-2}$, as expected from the standard $1/q^2$ divergence of the transverse susceptibility

$$\chi_\perp(q) = m_s^2 C_\perp(q)/T = \frac{m_s^2}{\rho_s q^2} \tag{4}$$

in the symmetry broken phase. In leading order, the longitudinal function is given by (c stands for "connected")

$$C_\parallel(r) \approx \frac{1}{4}\langle\vec{\Pi}^2(x)\vec{\Pi}^2(0)\rangle_c = \frac{N-1}{2}C_\perp^2(r), \tag{5}$$

and thus again decays algebraically as $1/r^{2(d-2)}$. Contrary to the naive mean field picture, where the longitudinal susceptibility $\chi_\parallel(q)$ below T_c is finite as $q \to 0$, the slow decay of $C_\parallel(r)$ implies that $\chi_\parallel(q \to 0) \sim T/\rho_s^2 q^{4-d}$ is divergent in $d < 4$ [4]. While the relation (5) is valid only to leading order in an expansion in powers of Π, the behavior $C_\parallel(r) \sim r^{-2(d-2)}$ is in fact expected to be exact at arbitrary temperatures below T_c [6], consistent with a rigorous correlation inequality for $N = 2$ [7].

To discuss the fluctuations of the spontaneous magnetization in a finite system at zero field, we define a fluctuating, extensive variable ($d1 = d^d x_1$)

$$\hat{M}_s = \frac{1}{V}\int d1 \int d2\, \vec{\psi}(1)\cdot\vec{\psi}(2), \tag{6}$$

with thermal average $\langle\hat{M}_s\rangle = V m_L^2$. Its fluctuations are then determined by the connected four point function

$$u_4(1234) = \langle\vec{\psi}(1)\cdot\vec{\psi}(2)\,\vec{\psi}(3)\cdot\vec{\psi}(4)\rangle_c. \tag{7}$$

Expanding this consistently up to order Π^4, one finds that in an infinite system this function may be expressed simply as

$$u_4(1234) = m_s^4\frac{N-1}{2}\cdot[C_\perp(r_{13}) - C_\perp(r_{14}) - C_\perp(r_{23}) + C_\perp(r_{24})]^2. \tag{8}$$

In order to obtain the scaling of $\mathrm{Var}\,\hat{M}_s$ in a finite system, we switch to a momentum representation and replace integrals $\int_q$ by discrete sums $V^{-1}\sum_q'$ over wave vectors. The $q = 0$ contribution is excluded since it describes an irrelevant global rotation of the order parameter $\vec{\psi}(x)$. It is then straightforward to show that

$$\mathrm{Var}\,\hat{M}_s = 2m_s^4(N-1)\left(\frac{T}{\rho_s}\right)^2\cdot\sum_q{}'\frac{1}{q^4}, \tag{9}$$

which is proportional to L^4 in $2 < d < 4$ by simple dimensional analysis [8]. Defining a numerical coefficient B by $\sum_q' q^{-4} = BL^4/8\pi^2$, we find that $B = 8E_3(2)/\pi^2 = 0.501$ for a 3D box with Dirichlet boundary conditions. Here $E_d(t) = \sum_{n_1=1,\ldots,n_d=1}^{\infty}(n_1^2 + \cdots + n_d^2)^{-t}$ is the generalized Epstein zeta function, convergent for $d < 2t$ (for periodic boundary conditions $B = 0.8375$ [3]).

Although our derivation of the general result (9) is based on an expansion in powers of Π and thus appears to be restricted to low temperature, the exponent in $\mathrm{Var}\,\hat{M}_s \sim L^4$ is universal below T_c [9] just like the $q^{-(4-d)}$ divergence of the longitudinal susceptibility. Similarly, the temperature dependence of $\mathrm{Var}\,\hat{M}_s$ will be $[T/\rho_s(T)]^2$ for arbitrary temperatures below T_c, vanishing proportional to T^2 at low temperatures since $\rho_s(T=0)$ is finite. This follows from the fact that, at any temperature T in the broken symmetry phase, the dominant finite size dependence is determined by the leading low energy constant in the effective field theory for fluctuations of the order parameter, which is precisely

ρ_s as defined in Eq. (1). New effective constants enter only in higher order terms, as discussed, for instance, in the context of chiral perturbation theory in QCD [10]. From Eq. (9), it follows that the relative fluctuations $\mathrm{Var}\,\hat{M}_s/\langle\hat{M}_s\rangle^2$ scale as $L^{-2(d-2)}$. The spontaneous magnetization is therefore self-averaging in $d > 2$ although weaker than expected naively, unless $d > 4$. In this context, it should be mentioned that, in the case of a non-Abelian symmetry $N \geq 3$, the spin stiffness ρ_s has anomalous fluctuations

$$\frac{\mathrm{Var}\,\rho_s}{\langle\rho_s\rangle^2} \sim (N-2)\cdot\left(\frac{T}{\rho_s L^{(d-2)}}\right)^2 \tag{10}$$

on its own, as shown by Chakravarty [11]. They are very similar to the fluctuations of the spontaneous magnetization discussed here and indeed arise from the same kind of Goldstone anomalies. A special situation appears in homogeneous superfluids ($N = 2$) at low temperatures. Since translation invariance requires the superfluid density to be equal to the full density n, the associated stiffness $\rho_s(T \to 0) = \hbar^2 n/m$ is independent of the interaction. Equation (9) then leads to the remarkable result that the relative fluctuations of the condensate number at low temperature ($d = 3$)

$$\lim_{T\to 0}\frac{\mathrm{Var}\,\hat{N}_0}{\langle\hat{N}_0\rangle^2} = \frac{B}{(n\lambda_T^2 L)^2} \tag{11}$$

are completely universal, depending only on density, system size, and the thermal wavelength $\lambda_T = h/\sqrt{2\pi m T}$, but not on whether the superfluid is weakly or strongly interacting [12]. Finally, it is important to realize that—as in the case of the Mermin-Wagner-Hohenberg theorem—the result (9) is independent of the order parameter dynamics. It applies equally to say ferromagnets or antiferromagnets even though the temperature dependence of the *average* order parameter $\langle\hat{M}_s\rangle$ is, of course, very different in both cases. This may be shown either by a microscopic derivation of Eq. (9) on the basis of noninteracting spin wave theory around a perfectly ordered ground state or, alternatively, by using a quantum mechanical generalization of the nonlinear σ model which describes magnons with dispersion either $\omega \sim q^2$ or $\omega \sim q$.

As shown above, the anomalous fluctuations (9) of the order parameter are a consequence of the $q^{-(4-d)}$ divergence of the longitudinal susceptibility in $d < 4$. In the following, we want to discuss the analog of this phenomenon in zero temperature phases where a continuous symmetry is broken. In this case, the specific dynamics of the order parameter is important. Here, we assume that the Goldstone modes have a linear spectrum $\omega = cq$ as in superfluids and antiferromagnets. An effective description of the ordered phase can then be obtained from a quantum mechanical nonlinear σ model, with a unit vector $\vec{\Omega}(x, \tau)$ which depends both on space x and imaginary time $\tau \in [0, \beta\hbar]$. The corresponding effective action

$$S[\Omega] = \frac{\rho_s}{2\hbar}\int_0^{\beta\hbar} d\tau \int d^d x\left[[\nabla\vec{\Omega}(x)]^2 + \left(\frac{1}{c}\partial_\tau\vec{\Omega}\right)^2\right] \tag{12}$$

has the spin wave velocity c as the only additional parameter. Together with the renormalized value m_s^2 of the long range order, ρ_s and c completely determine the low energy properties of the ordered phase. As an example, this model applies both to the Néel ordered state of 2D quantum antiferromagnets discussed extensively in the context of high temperature superconductors [13] or to the superfluid phase of cold atoms in an optical lattice which has been realized recently in 3D [14]. Using again a lowest order expansion in powers of the small fluctuations $\vec{\Pi}(x, \tau)$ in the standard decomposition $\vec{\Omega} = (\vec{\Pi}, \sqrt{1-\vec{\Pi}^2})$, the transverse correlation function at $T = 0$ in two dimensions is given by [15]

$$C_\perp(x, \tau) = \frac{\hbar c}{4\pi\rho_s}\frac{1}{\sqrt{r^2 + (c\tau)^2}}. \tag{13}$$

Analytic continuation to real time $t = -i\tau$ and Fourier transformation give the standard form of the transverse dynamical susceptibility in any dimension,

$$\chi_\perp(q, \omega) = \frac{m_s^2}{\rho_s}\frac{1}{q^2 - \omega^2/c^2}. \tag{14}$$

It leads to the expected quasiparticle pole at $\omega = cq$, reflecting the presence of undamped antiferromagnetic spin waves. Considering the longitudinal correlations, the analog of the relation (5) again applies to leading order. As a result, the longitudinal dynamical susceptibility in 2D turns out to be

$$\chi_\parallel(q, \omega) = \frac{(N-1)m_s^2\hbar c}{16\rho_s^2}\frac{1}{\sqrt{q^2 - \omega^2/c^2}}, \tag{15}$$

which has a branch cut rather than a simple pole. Formally the result (15) is completely analogous to the $1/q$ divergence of the classical static susceptibility $\chi_\parallel(q)$ in $d = 3$ discussed above. Indeed the effective dimensionality of the 2D quantum antiferromagnet is $d + z = 3$ and the dependence on frequency ω is dictated by the formal Lorentz invariance of the model (12).

In order to relate these results to directly observable quantities, we consider the rotationally averaged staggered susceptibility (for antiferromagnets $N = 3$)

$$\chi_s(q, \omega) = \frac{N-1}{N}\chi_\perp(q, \omega) + \frac{1}{N}\chi_\parallel(q, \omega). \tag{16}$$

For frequencies $\hbar\omega \gg T$, its imaginary part is equal to the dynamic structure factor $S(q, \omega)$ at $\omega > 0$ up to a factor $2\hbar$, giving

$$S(q, \omega) = 2m_s^2\xi_J\frac{N-1}{N}\left[\frac{\pi}{2q}\delta(\omega - cq) + \frac{\xi_J}{16}\frac{\theta(\omega - cq)}{\sqrt{\omega^2 - c^2q^2}}\right]. \tag{17}$$

The longitudinal fluctuations of the Néel order thus lead to a critical continuum above the spin wave pole at $\omega = cq$, which decays only algebraically. The continuum results from the decay of a normally massive amplitude mode with momentum $\vec{p}$ into a pair of spin waves with momenta $\vec{q}$ and $\vec{p} - \vec{q}$, which is possible for any $\omega > cq$, with a singular cross section because of the large phase space. The amplitude mode is thus completely overdamped in two dimensions [5]. The relative weight of these fluctuations compared to the dominant transverse contribution is determined by the dimensionless parameter $q\xi_J$, with $\xi_J = \hbar c/\rho_s$ the Josephson correlation length. This length controls the crossover from the Goldstone regime $q\xi_J \ll 1$, where the spin dynamics is well described by small fluctuations around a Néel ordered ground state to quantum critical fluctuations at $q\xi_J \gg 1$. The quantum critical regime has a dynamical susceptibility of the form [16]

$$\chi_s(q, \omega) = \frac{m_s^2}{\rho_s}\left(\frac{N\xi_J}{2\pi}\right)^{\eta} \cdot \frac{A_Q}{(q^2 - \omega^2/c^2)^{1-\eta/2}}, \qquad (18)$$

with an amplitude A_Q close to 1 and an exponent η which is nearly zero. Now, in the regime where the ground state exhibits strong Néel order, the Josephson correlation length is only several lattice constants [13], and thus the quantum critical regime is hardly accessible. In turn, there is a rather wide Goldstone regime characterized by $q\xi_J \ll 1$ and $q\xi(T) \gg 1$, where $\xi(T)$ is the finite 2D correlation length over which Néel order is lost at finite temperature. Because of the exponential dependence [13]

$$\xi(T) \sim \xi_J \exp(2\pi\rho_s/T), \qquad (19)$$

the relevant range of wave vectors $\xi^{-1}(T) < q < \xi_J^{-1}$ is rather wide at low temperatures. Experimentally, spin waves in 2D quantum antiferromagnets have been observed by inelastic neutron scattering in the vicinity of the Néel ordering wave vector (π, π). In a constant ω scan, the first term in (17) gives rise to a sharp peak at $q = \omega/c$ with an amplitude $\sim 1/q$ [17]. The second contribution in (17) due to the longitudinal fluctuations implies an additional algebraic tail towards smaller wave vectors, provided q is in the Goldstone regime. At room temperature, where $\xi(T)$ is several hundred Å and with typical values of ξ_J, this gives a range $0.005 < q(\text{Å}^{-1}) < 0.05$. Given the resolution in Ref. [17], detection of this algebraic tail appears very difficult; however, high resolution measurements may be able to observe the additional contribution from the longitudinal spin fluctuations, which apparently behave similar to the critical fluctuations (18), however, with a rather large exponent $\eta = 1$.

In summary, it has been shown that Goldstone modes associated with a broken continuous symmetry lead to fluctuations of the zero field order parameter which scale proportional to L^4 at any temperature below T_c. For 2D ordered ground states like the Néel phase of high temperature superconductors, the underlying longitudinal fluctuations lead to an additional critical continuum beyond the simple spin wave pole in the dynamic structure factor, whose detection requires high resolution neutron scattering experiments.

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