Onset of Collective and Cohesive Motion

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We study the onset of collective motion, with and without cohesion, of groups of noisy self-propelled particles interacting locally. We find that this phase transition, in two space dimensions, is always discontinuous, including for the minimal model of Vicsek *et al.* [Phys. Rev. Lett. **75**,1226 (1995)] for which a nontrivial critical point was previously advocated. We also show that cohesion is always lost near onset, as a result of the interplay of density, velocity, and shape fluctuations.

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Collective motion can be observed at almost every scale in nature, from the familiar human crowds [1], bird flocks, and fish schools [2] to unicellular organisms such as amoebae and bacteria [3], individual cells [4], and even at a microscopic level in the dynamics of actin and tubulin filaments and molecular motors [5,6]. Whereas biologists tend to build detailed representations of a particular case, the ubiquity of the phenomenon suggests underlying universal features and thus gives weight to the bottom-up modeling approach usually favored by physicists [7].

In this respect, the simple model introduced by Vicsek and collaborators [8] stands out because of its minimal character and the *a priori* least-favorable conditions in which it is defined. In the Vicsek model (VM), identical pointwise particles move at constant velocity and interact locally by trying to align their direction with that of neighbors. Remarkably, even in the presence of noise and in the absence of leaders and global forces, orientational long-range order arises, i.e., collective motion emerges, if the density of particles is high enough or, equivalently, if the noise is weak enough. The existence of the ordered phase was later "proved" by a renormalization-group approach based on a phenomenological mesoscopic equation [9]. More recently, this work was extended to the case where the ambient fluid is taken into full account, yielding novel mesoscopic equations for suspensions of self-propelled particles [10].

The nature of the nonequilibrium phase transition to collective motion, however, is not well established. Vicsek *et al.* concluded from numerical simulations in two and three dimensions that it is continuous ("second order") and characterized by a set of critical indices, but these results remain somewhat crude [11], even though the undeniably minimal character of the VM makes it a good candidate for representing a universality class.

Moreover, from a modeling point of view, an often desirable ingredient missing in the VM is *cohesion*: When put together in an infinite space, particles do not stay together and fly apart. In other words, no collective motion is possible in the zero-density limit of the VM. Recently, we have shown how one can ensure cohesion in simple models derived from the VM without resorting to leader particles or long-range or global forces [12].

In this Letter, we study the onset of collective motion with and without cohesion in this very general setting, trying to assess the universality of the results of Vicsek *et al.* In both cases, we find that the onset of collective motion in the VM and related models is actually *discontinuous* ("first order") and that its apparent continuous character is due to strong finite-size effects. We also show that without cohesion the transition point is nevertheless accompanied by a nontrivial superdiffusive behavior of particles which, we argue, could be measured experimentally. In the presence of cohesion, our study reveals that the onset of collective motion is the theater of a complex interplay between density, velocity, sound, and shape modes, giving rise to fascinating dynamics.

The original VM is defined as follows: Identical pointwise particles move synchronously at discrete time steps $\Delta t = 1$ by a fixed distance v_0. In two space dimensions—to which we restrict ourselves in the following—the direction of motion of particle j is just an angle θ_j, calculated from the previous directions of all particles k within an interaction range $r_0 = 1 > v_0 \Delta t$:

$$\theta_j^{t+1} = \arg\left[\sum_{k\sim j} e^{i\theta_k^t}\right] + \eta \xi_j^t, \qquad (1)$$

where ξ_j^t is a delta-correlated white noise ($\xi \in [-\pi, \pi]$). This introduces a tendency to align with neighboring particles, with two simple limits: In the absence of noise, interacting particles align perfectly, quickly leading to complete orientational order. For maximal noise ($\eta = 1$), particles follow random walks. The transition that necessarily lies in between these two regimes can be characterized by the following instantaneous order parameter:

$$\varphi^t \equiv \frac{1}{N}\left|\sum_{j=1}^{N} e^{i\theta_j^t}\right|, \qquad (2)$$

where N is the total number of particles.

Varying either the noise strength η or the particle density $\rho = N/L^2$ in periodic domains of linear size L, Vicsek *et al.* found that $\langle\varphi\rangle$ varies continuously across the transition, suggesting the existence of a critical point [8]. Studying finite-size effects, they estimated a set of scaling exponents. Interested in assessing the universality of these results and possibly improving these estimates, we first introduced simple modifications of the original VM such as changing v_0 or adding a repulsive force between particles to give them a finite extent. Using the finite-size scaling ansatz appropriate for XY-model-like systems, domain sizes, and particle numbers similar to those used in [8], but with much better, well-controlled statistics, we were only able to estimate a roughly coherent set of critical exponents after allowing for rather strong corrections to scaling [13].

For modeling reasons, we also changed the way noise is incorporated in the system. In (1), particles make an error when trying to take the new direction they have perfectly calculated ("angular noise"). One could argue that, rather, errors are made when estimating the interactions, for example, because of a noisy environment. This leads to change Eq. (1) into, e.g.,

$$\theta_j^{t+1} = \arg\left[\sum_{k\sim j} e^{i\theta_k^t} + \eta n_j^t e^{i\xi_j^t}\right], \tag{3}$$

where n_j^t is the current number of neighbors of particle j. In this case of "vectorial noise," the onset of collective motion is *discontinuous*: For large-enough system sizes, $\langle\varphi\rangle$ jumps abruptly to zero as η is decreased, whereas it varies smoothly in the original VM [Fig. 1(a)]. This is perhaps best seen from the behavior of the so-called Binder cumulant $G = 1 - \langle\varphi^4\rangle/3\langle\varphi^2\rangle^2$ [Fig. 1(b)]. In the case of vectorial noise, G falls to negative values near η_c, the sign of a discontinuous transition, together with the phase coexistence expected then.

Going from angular to vectorial noise is indeed a less innocent modification than those mentioned earlier: In model (3), locally ordered regions are subjected to weaker noise than disordered ones. However, it was unclear to us what precisely would be the mechanism to change the order of the transition upon introducing this nonlinear term. Considering in addition the strong corrections to scaling found with angular noise, we strived to reach larger system sizes in some of these cases, albeit at the cost of statistical accuracy [13]. The conclusion of these numerical efforts is that the transition is discontinuous in *all* cases, with finite-size effects being somewhat weaker at low densities. As an example, the behavior of G with increasing system size shown in Fig. 2(a) for the original VM at $\rho = \frac{1}{8}$ reveals the characteristic fall to negative values. The distribution function of φ^t is bimodal around threshold, without any intermediate unimodal regime [Fig. 2(c)]. Thus, the continuous transition reported by Vicsek *et al.* is only apparent.

In the ordered phase, the particles are organized in density waves moving steadily in a disordered "vapor pressure" background of well-defined asymptotic density [Fig. 2(b)]. These solitary waves become metastable to a long-wavelength longitudinal instability below the density threshold η_c (defined to be located at the minimum of G), leading to a hysteresis loop. At threshold and below, the disordered phase consists of nucleated ordered patches competing in space and time [Fig. 2(d)].

At threshold, in the disordered phase, a universal nontrivial algebraic scaling law is nevertheless found: The superdiffusive behavior of particles already reported by us in [14] is valid in all cases. Trajectories then consist of

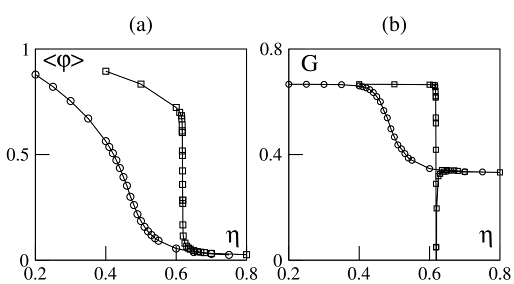

FIG. 1. Onset of collective motion in cohesionless models (1) (original VM, circles) and (3) (vectorial noise, squares). Variation of order parameter φ (a) and Binder cumulant G (b) with the noise strength η. ($v_0 = 0.5$, $L = 32$, $\rho = 2$, and equivalent statistics for both models.)

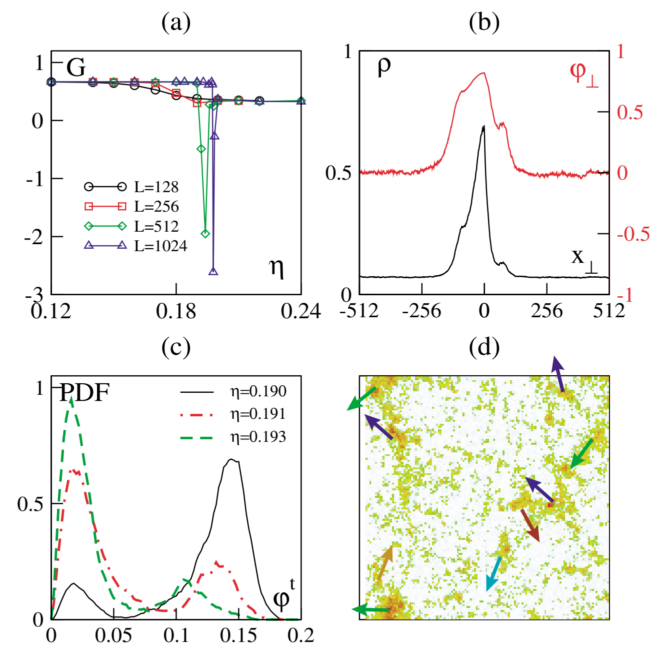

FIG. 2 (color online). Discontinuous character of the onset of collective motion in the original VM at $\rho = \frac{1}{8}$. (a) G vs η at various system sizes. (b) Transverse density (bottom curve) and order-parameter profile (top curve) in the ordered phase ($L = 1024$, $\eta = 0.18$) (c) probability distribution function (PDF) of φ^t near the transition point, $t \in [\tau; 500\tau]$; here the correlation time is $\tau \simeq 10^5$ [13], $L = 512$. (d) Snapshot of coarse-grained density field in disordered phase at threshold, $\rho = 2$, $L = 256$. The arrows indicate the direction of motion of dense, ordered regions.

"flights," occurring when a particle is caught in a moving ordered patch, separated by normal diffusion in the disordered regions. The mean square displacement of particles $\langle \delta r^2 \rangle$ varies similar to t^α with $\alpha = 1.65(5)$.

We now turn to the onset of collective motion in the presence of cohesion. As shown in [12], the cohesion of a population of particles can be maintained without resolving to long-range or global interactions. In the spirit of the VM, and following [15], a two-body short-range interaction force competing with the alignment tendency is introduced, leading to the model

$$\theta_j^{t+1} = \arg\left[\alpha \sum_{k\sim j} e^{i\theta_k^t} + \beta \sum_{k\sim j} f_{jk}^t e^{i\theta_{jk}^t} + \eta n_j^t e^{i\xi_j^t}\right], \quad (4)$$

where α and β control the strength of alignment and cohesion, and θ_{jk}^t is the direction of the vector linking particle j to particle k. The interaction force between these two particles, of amplitude f_{jk}^t, is actually repulsive up to an intermediate equilibrium distance r_e, with a short-range hard core at r_c and attractive up to the interaction range r_0. In the following, as in [12], we used

$$f_{jk} = \begin{cases} -\infty & \text{if } r_{jk} < r_c \\ \frac{1}{4}\frac{r_{jk}-r_e}{r_a-r_e} & \text{if } r_c < r_{jk} < r_a \\ 1 & \text{if } r_a < r_{jk} < r_0, \end{cases} \quad (5)$$

where r_{jk} is the distance between j and k, with $r_c = 0.2$, $r_e = 0.5$, and $r_a = 0.8$. Note that vectorial noise was chosen in (4) in the hope of reaching asymptotic properties more easily.

The above model has three main parameters, α, β, and η, only two of which are independent. The phase diagram in the (α, β) plane (with $\eta = 1$ fixed arbitrarily) was presented in [12], where, moreover, only neighbors in the Voronoi sense are considered in the sums of (4). For large enough β, cohesion is maintained, even in the zero-density limit. This "gas/liquid" transition is followed, at larger β values, by the onset of positional (quasi) order, i.e., a "liquid/solid" transition. For large α, these liquid or solid cohesive groups move, whereas they remain static (up to finite-size fluctuations) for small α.

In the "liquid case" (intermediate β values), the onset of motion is accompanied by a loss of cohesion: While small groups set in motion smoothly without breaking up [Fig. 3(a), dashed lines], larger groups gradually subdivide into several parts of roughly equivalent size linked by filamentary structures, in contrast with their more compact shapes before and after onset (Fig. 4). The filaments themselves are quite static [Fig. 4(d)] but are displaced by the subgroups which move coherently so that they eventually break up, as indicated by the dip in the normalized largest connected cluster size n/N in Fig. 3(a). Increasing α, large groups follow the same precursor of the transition as smaller groups, but when their fragmentation occurs the order parameter falls back, leaving an intermediate peak [around $\alpha = 1.7$ in Fig. 3(a)].

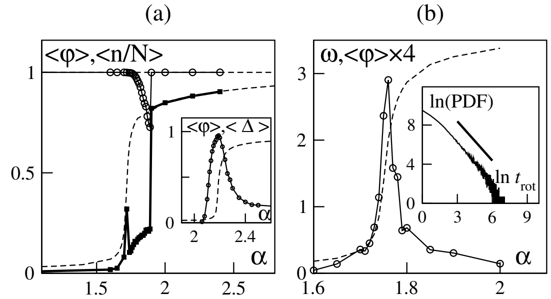

FIG. 3. Onset of motion of cohesive groups in model (4) with $\eta = 1$, $v_0 = 0.05$. (a) $\langle\varphi\rangle$ and n/N (normalized size of largest connected cluster) vs α [$\rho = \frac{1}{16}$, $\beta = 20$ (liquid phase): dashed lines, $N = 4096$; solid lines, $N = 16\,384$]. Inset: solid group ($\beta = 84$) of $N = 4096$ particles: dashed line, $\langle\varphi\rangle$; solid line, relative diffusion of initially neighboring particles $\Delta \equiv \langle \frac{1}{n_j}\sum_{k\sim j}[1 - r_{jk}^2(t)/r_{jk}^2(t+T)]\rangle_{j,t}$, where $T \approx 20N$ ($\Delta \simeq 1$ in the liquid phase, while $\Delta \simeq 0$ in the solid phase, see [12]). (b) Variation with α of the maximal absolute rotation angle $|\omega|$ averaged over 100 samples of 1000 vortices [$N = 2048$, $\rho = \frac{1}{32}$, $\beta = 30$ (liquid phase)]. Dashed line: $\langle\varphi\rangle$ during the same runs. Inset: Distribution of rotation times at the transition with decay exponent $\sim$1.3.

Increasing α further, $\langle\varphi\rangle$ rises again and finally jumps to $\langle\varphi\rangle = 1$ when full cohesion is recovered [for $\alpha = 1.88$ in Fig. 3(a)]. This discontinuous jump is the true location of the transition: For an infinite group, the onset of motion must occur abruptly near this value, as the precursory features described above disappear because the population divides into infinitely many subgroups whose influences average themselves out. Meanwhile, cohesion is only lost *at* the transition point in this asymptotic picture.

The breakup of large cohesive groups around threshold is probably closely related to what happens in the case without cohesion: The subgroups connected by filaments may correspond to the ordered patches seen in the disordered phase near threshold in Fig. 2(d). The breakup itself can be seen as resulting from the maximal effect of acoustic modes on the shape of the group [13]. Also affecting the shape dynamics are rotational modes: The subgroups seen in Fig. 4(a) not only move but they also rotate slowly [16]. Rotation is not steady, but intermittent. We recorded the rotation times and their corresponding angles. Extremal statistics analysis reveal that the tendency to rotate is maximal at the onset of motion [Fig. 3(b)]. Moreover, at threshold, the distribution of rotation times is algebraic with a decay exponent such that it has no finite mean [inset of Fig. 3(b)].

The onset of motion of the "solid" groups (large β values) is accompanied by a loss of positional order: These crystals melt near the transition [inset of Fig. 3(a)]. Given the above results in the liquid case, one can expect very large solid groups to melt and then subdivide and lose cohesion in the transition region, making the onset of motion asymptotically discontinuous.

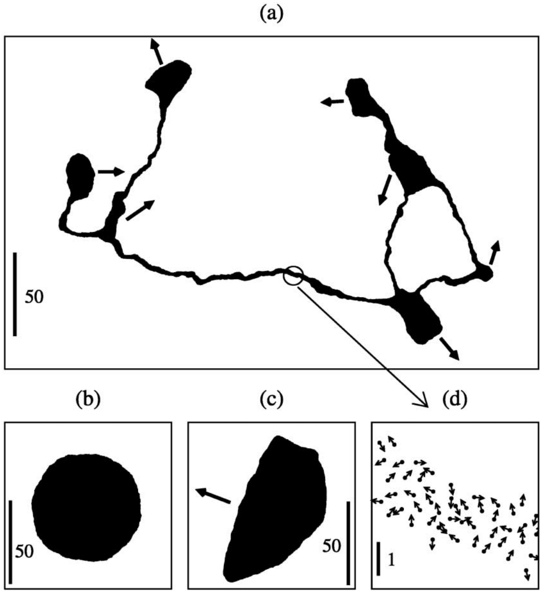

FIG. 4. Typical shape of a liquid cohesive group of 16 384 particles [model (4), $\rho = \frac{1}{16}$, $\beta = 20$ arrows indicate direction of motion]. (a) At onset before loss of cohesion, $\alpha = 1.78$. (b) Static phase, round shape, $\alpha = 0.5$. (c) In moving phase, typical triangular form (see [12]), $\alpha = 2.5$. (d) Closeup of a filament; no local order is apparent.

In summary, the onset of collective motion in the VM as well as in related models with and without cohesion is always discontinuous, and the critical behavior reported in [8] is only apparent and due to (strong) finite-size effects. Without cohesion, the ordered phase consists in density waves propagating steadily in a disordered background. With a short-range repulsion/attraction interaction, the cohesion ensured both in the disordered and ordered phases is broken at the onset of motion under the competing influence of sound, density, and shape modes. The resulting mesoscopic subgroups are linked by filaments which, however, we believe to be probably nonuniversal, model-dependent structures.

At the theoretical level, ongoing work is directed towards the understanding of the complex interplay between shape (surface tension) and acoustic modes, and of the stability properties of the density waves. At the experimental level, it remains difficult to study quantitatively bird flocks and fish schools, and, moreover, we have no specific prediction as to the onset of motion of these cohesive groups [17]. Without cohesion, however, the universal superdiffusive behavior observed in the disordered phase near threshold could be observed experimentally. As already suggested in [14], bacteria such as *E. Coli* might be good self-propelled particles. Human melanocytes also look promising in this respect as shown remarkably by the group of Gruler [4]. Finally, "motility assays" consisting of grafted molecular motors such as kinesin (respectively, myosin) moving filaments made of tubulin (respectively, actin) might provide the simplest setting in which to investigate superdiffusion at onset, given the available observation techniques [5,6].

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