Fluctuations Do Matter: Large Noise-Enhanced Halos in Charged-Particle Beams

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The formation of beam halos has customarily been described in terms of a particle-core model in which the space-charge field of the oscillating core drives particles to large amplitudes. This model involves parametric resonance and predicts a hard upper bound to the orbital amplitude of the halo particles. We show that the presence of colored noise due to space-charge fluctuations and/or machine imperfections can eject particles to much larger amplitudes than would be inferred from parametric resonance alone.

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Beam loss is a major concern for high-current light-ion accelerators such as are needed to drive high-intensity spallation neutron sources [1]. Just a tiny impingement, $\sim$1 W/m, could generate radioactivation that would preclude routine, hands-on maintenance [2]. For a 1 mA, 1 GeV light-ion beam, i.e., for baseline beam parameters of the Spallation Neutron Source (SNS) presently under construction [3], this criterion translates to just one in 10^6 particles lost per meter, a quantity that scales linearly with average beam current. Accordingly, a comprehensive understanding of beam-halo formation is imperative.

Early efforts to identify the fundamental mechanisms of halo formation centered on the use of a "particle-core" model [4–6]. The basic recognition was that, if a uniform-density core is made to pulsate, particles that initially lay outside the core and that resonate with its pulsations could reach large amplitudes and form a "halo." This led to the identification of parametric resonance as the essential mechanism of halo formation. A key feature of parametric resonance in the context of the particle-core model is a hard upper bound to the amplitude that a halo particle can reach [5]. Because the particle's orbital frequency is a function of its amplitude, at sufficiently large amplitude the particle falls out of resonance with the core and thereby its amplitude ceases from growing further. The prospect that the beam halo is "self-collimating" has led to hope that aperture requirements for beam-line components might be modest. Smaller apertures are preferred in that, for example, they favor higher-efficiency operation of the accelerating cavities. In turn, a large body of literature has developed over the past ten years concerning the putative maximum halo amplitude, e.g., [7–13].

One feature that is unavoidable in real accelerators but is commonly overlooked in simulations is the presence of noise. The noise will manifest itself by way of the electromagnetic fields external to the beam, which then self-consistently influence the beam's evolving space-charge potential. Noise sources could include hardware irregularities that establish fluctuating image-charge forces, jitter in power supplies, misalignments and/or asymmetries of beam line components, etc. In the context of simulations, it could also include details in the space-charge potential that the simulation cannot model precisely. Moreover, the noise will generally comprise a superposition of "colored" noise, i.e., that for which the autocorrelation time is nonzero. For example, the autocorrelation time of noise in the collective space-charge potential could be short, say of the order of a plasma period, whereas for hardware irregularities/misalignments it could be long, say several betatron (orbital) periods. Herein, by generalizing simple particle-core models, we show that the presence of colored noise can potentially boost statistically rare particles to ever-growing amplitudes by continually kicking them back into phase with the core oscillation.

Following the ground-breaking work that introduced the particle-core model [5,6], we consider particles on radial orbits through an infinitely long, axially symmetric, space-charge-limited (zero tune depression) beam "core" pulsating at a single frequency due to an imbalance, i.e., mismatch, between the repulsive, collective space-charge force and the confining external focusing force. Upon linearizing the beam-envelope equation in terms of the core-oscillation amplitude, one finds the solution $R(t) \simeq R[1 + (M - 1)\cos\omega t]$, wherein ω is the core-oscillation angular frequency and $M = R(0)/R$ is the mismatch parameter, i.e., the ratio of initial-to-matched core radii. For the space-charge-limited beam $\omega = \sqrt{2}\Omega$, where Ω denotes the external focusing angular frequency. We fold this harmonically oscillating core into a dimensionless equation of particle motion:

$$\ddot{x} + \left[1 - \frac{\Theta(1 - |x|)}{[1 + (M - 1)\cos\omega t]^2} - \frac{\Theta(|x| - 1)}{x^2}\right]x = 0. \tag{1}$$

The transverse coordinate x is normalized to the radius R of the matched beam; time is multiplied by Ω which means all frequencies are expressed as multiples of Ω;

and $\Theta(u)$ is the Heaviside step function. This model, henceforth called "Model I," is simpler and computationally less expensive than the original particle-core model [5] but preserves its vital parametric resonance.

Because Model I is strictly one dimensional and contains a step-function discontinuity, we also study a second model for which the unperturbed beam is a spherically symmetric configuration of thermal equilibrium (TE) [14]. Particles orbit in this model, henceforth called "Model II," according to

$$\begin{aligned} \ddot{\mathbf{x}} &= -\nabla\Psi, & \Psi &= \Psi_0 + \Psi_1, \\ \Psi_0 &= \tfrac{1}{2}\Omega^2 r^2 + \Phi(r), & \Psi_1 &= \mu\Phi(r_1)\sin\omega t, \\ r^2 &= x^2 + y^2 + z^2, & r_1^2 &= 0.8(x^2 + y^2) + z^2, \end{aligned} \tag{2}$$

in which the external focusing angular frequency is $\Omega = 1.0001/\sqrt{3}$. As explained in Ref. [14], the coordinates and time are measured in units of Debye length and inverse plasma angular frequency, so the normalization differs from that of Model I. The self-potential $\Phi(r)$ corresponds to "intermediate space charge" (0.36 tune depression [15]). The potential Ψ_1 is a prolate spheroidal perturbation whose strength corresponds to the parameter μ.

To Models I and II we add fluctuations in the form of Gaussian colored noise such that $\omega \to \omega(t) = \omega + \delta\omega(t)$, with $\delta\omega(t)$ sampling an Ornstein-Uhlenbeck process. Its first two moments fully determine the statistical properties of the noise: $\langle\delta\omega(t)\rangle = 0, \langle\delta\omega(t)\delta\omega(t_1)\rangle \propto \exp(-|t - t_1|/t_c)$, in which t_c denotes the autocorrelation time. Though we add the noise to the core-oscillation frequency, we also confirmed that adding colored noise to the external focusing frequency does not significantly change the results.

After generating a colored-noise signal using an algorithm first presented in Ref. [16], we calculate $\langle|\delta\omega|\rangle$ which becomes a measure of the noise strength. The influence of noise on halo formation should in principle depend on its strength and its autocorrelation time. For two choices of autocorrelation time, $t_c = 1.5\tau$ and 12τ, τ denoting the orbital period of a typical halo particle, we investigated a broad range of strengths, specifically $10^{-5} \leq \langle|\delta\omega|\rangle \leq 1$, to see to what extent the results may be regarded as generic. Manifestations of colored noise that a particle might feel are illustrated in Fig. 1.

In a real beam each individual particle will have its own distinct initial conditions and thus experience a manifestation of the noise that differs from that seen by each of the other particles. For example, in the axisymmetric Model I, each particle initially occupying a thin annulus centered at radius $x(0)$ will experience noise differing from that seen by each of the other particles initially in that annulus because the particles start at different angular coordinates. The same is true for particles initially occupying a spherical shell centered on radius $r(0)$ in Model II. Accordingly, we adopted a "survey strategy." Upon choosing initial conditions $x(0)$ and

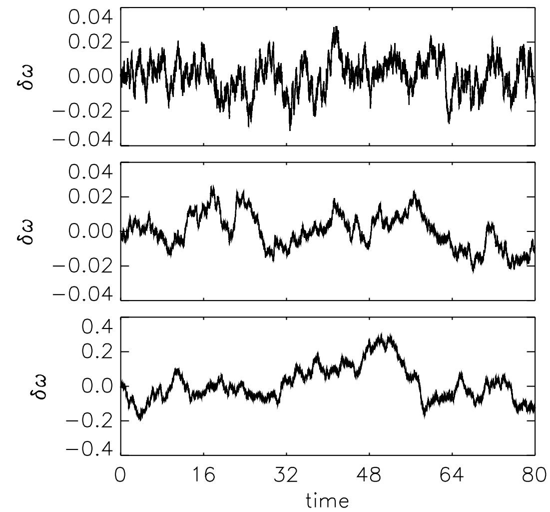

FIG. 1. Example manifestations of colored noise along an orbit for $\langle|\delta\omega|\rangle = 0.01$ and for which $t_c = 1.5\tau$ (top), 12τ (center), and for $\langle|\delta\omega|\rangle = 0.1$ with $t_c = 12\tau$ (bottom).

$r(0)$ for Eqs. (1) and (2), respectively, and for a specific choice of noise parameters, we sequentially computed 10 000 orbits, each experiencing its own random manifestation of the colored noise, and we cataloged the maximum amplitudes of these orbits. We set the initial conditions of the orbit in Model I at $x(0) = 1.20$, $\dot{x}(0) = 0$, and in Model II at $r(0) = 1.23$, $\dot{r}(0) = 0$. In the unperturbed TE sphere of Model II, and for realistic proton beam parameters, there are $\sim 4 \times 10^9$ particles per bunch, i.e., ~ 0.6 nC [17]. There are $\sim 3 \times 10^4$ particles in the range $r = 1.23 \pm (0.5 \times 10^{-4})$, a thin spherical shell centered on $r(0)$, and located well into the Debye tail of the bunch. Accordingly, the chosen sample size is realistic.

We computed the orbits using a fifth-order Runge-Kutta integrator with variable time step [18]. We chose the initial time step to be 10^{-2}; the fractional error in the orbital coordinates and momenta was within 10^{-9} for Model I and 10^{-7} for Model II. Smaller initial time steps yielded only tiny changes in the results that were attributable to the statistical nature of the simulations.

For Model I we chose $\omega = \sqrt{2}$ and computed orbits from Eq. (1) first without, then with, colored noise; we found that different choices of ω do not change the essential findings. For zero noise, the maximum orbital amplitude $|x_{\max}|$ does have a hard upper bound in keeping with parametric-resonance arguments.

For specified noise parameters, we focus on the one particle out of the sample of 10 000 that reaches the largest amplitude during the integration time of 80τ, a time that is representative of the transit time through a 1 GeV proton linac. Results for $M = 1.1, 1.3, 1.5$, and fixed $t_c = 12\tau$ are provided in Fig. 2, in which $|x_{\max}|$ versus $\langle|\delta\omega|\rangle$ is plotted. The figure also shows the average $|x_{\max}|$ reached by particles in the sample. One sees that, over a broad range of noise strengths, rare particles are ejected

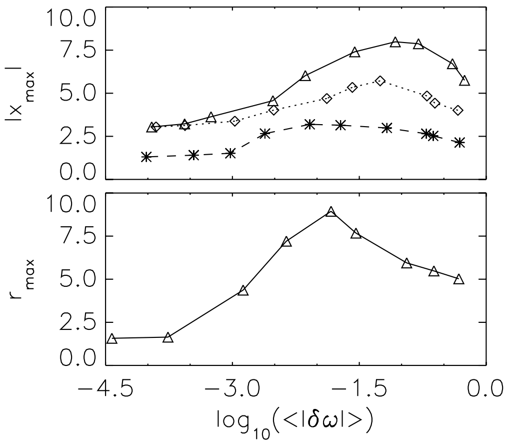

FIG. 2. (top) Largest orbital amplitude in Model I with $M = 1.1$ (*), 1.3 (◇), 1.5 (△), and $t_c = 12\tau$ plotted against $\langle|\delta\omega|\rangle$. With zero noise, the orbital amplitudes are 1.20, 2.54, and 2.92 for $M = 1.1$, 1.3, and 1.5, respectively. (bottom) Same for Model II with perturbation parameter $\mu = 0.5$. With zero noise, the maximum orbital amplitude is 1.36.

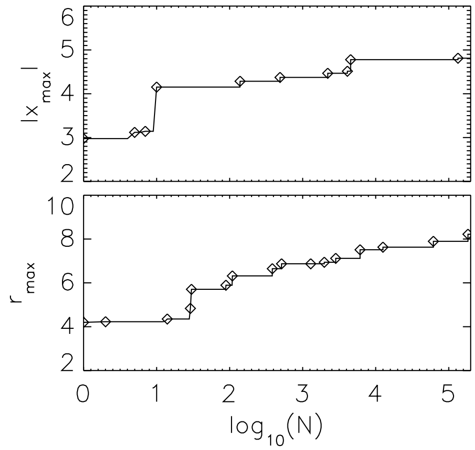

FIG. 3. Maximum orbital amplitude vs sample size N with $\langle|\delta\omega|\rangle = 0.004$ and $t_c = 12\tau$ for Model I with $M = 1.5$ (top), and Model II (bottom).

to larger amplitudes relative to parametric resonance alone. For example, a mere 1% fluctuation in the core-oscillation frequency roughly doubles the maximum amplitude reached compared to the case of zero noise. Interestingly, we found for $t_c = 1.5\tau$ that the results are very similar. Also, the curves each show a local maximum; very weak noise has little effect, and very strong noise results in large kicks that inhibit resonant coupling between halo particles and the core oscillation.

For Model II, we chose $\mu = 0.5$, corresponding to a root-mean-square mismatch $\simeq 1.3$, and $\omega = 1.7$, a completely arbitrary driving frequency. Results appear in the bottom panel of Fig. 2; different choices of ω do not change the essential findings. Models I and II are distinctly different, yet the influence of the noise on the maximum orbital amplitudes is qualitatively similar. This is a noteworthy finding in that we constructed Model II *ad hoc*, with no predisposition toward matching the results of Model I. Accordingly, the role of colored noise in generating large distended halos in time-dependent potentials would seem to be generic.

If the number of particles in the sample is increased with all else being the same, then the largest amplitude reached by the single special particle increases. As Fig. 3 indicates, once the sample size is sufficiently large, the maximum amplitude scales quasilogarithmically with sample size. This trend, heretofore unexplained, was seen in massive parallelized beam-dynamics simulations of an earlier design of the SNS linear accelerator that included various machine imperfections [19]. In runs involving 10^4, then 10^5, then 10^6, then 10^7 simulation particles, the maximum halo extent increased, but it seemed to approach a limiting value with runs above 10^8 particles. Inasmuch as these runs were self-consistent, the phenomenology they reflect is suggestive of the influence of increasingly fine resolution of details in the potential that are beyond the scope of a simple particle-core model. They also exemplify that a large number of particles is needed to discern the impact of these details on halo formation and structure.

If the integration time is extended indefinitely, as might be physically representative of a storage ring, for example, then there are statistically rare orbits that continue to grow to seemingly unlimited amplitudes. Examples of such orbits in Models I and II appear in Fig. 4. These long-time orbits exemplify that there is in principle no hard upper bound to the halo amplitude in the presence of colored noise.

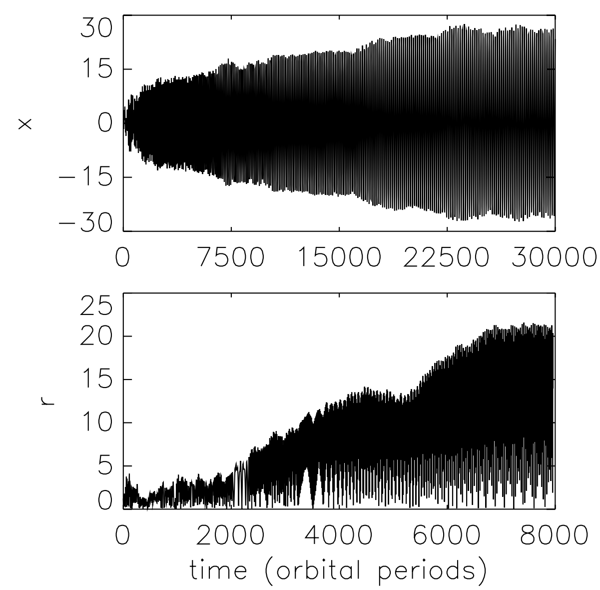

FIG. 4. Long-time evolution of a large-amplitude orbit given noise with $\langle|\delta\omega|\rangle = 0.01$, $t_c = 12\tau$ for Model I with $M = 1.5$ (top) and Model II (bottom).

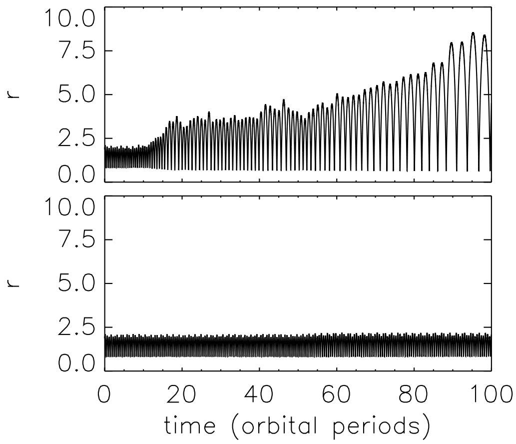

FIG. 5. (top) Example of a rare large-amplitude orbit in the Plummer potential; $\mu = 0.05$, $\langle|\delta\omega|\rangle = 0.02$, and $t_c = 12\tau$. (bottom) Particle orbit with zero noise.

It remains to explore further the extent to which this phenomenology applies in real machines. Doing so will involve further simulations of beams in real beam lines; as we have seen, machine imperfections will matter. One possibly fruitful approach is to extract the coarse-grained, time-dependent potential from the simulations and then add noise and pursue a statistical analysis of test particles in parallel to what we have done here. Alternatively, the colored noise may be built directly into the simulation itself, although the simulation will then need to incorporate a sufficiently large number of particles to furnish enough statistics on the halo population. A realistic manifestation of the colored noise would need to reflect the machine design, i.e., by properly including imperfections in the fields and hardware alignment, and details of the evolving space-charge potential such as a sufficiently detailed mode spectrum. Of course, as the beam is accelerated, space charge and its attendant parametric resonances become weaker, and their influence on the halo decreases.

As a relevant aside, we also analyzed this mechanism in the context of a self-gravitating stellar system for which environmental noise from surrounding galaxies will self-consistently influence the dynamics. Specifically, we considered a perturbed Plummer model, a configuration for which the unperturbed collective potential scales as $(1 + r^2/3)^{-1/2}$ [20], and we applied the same procedure described herein for Model II. Though it is a restoring force, gravity is so weak that, combined with the noise, only a relatively tiny oscillatory perturbation suffices to pump stars to very large amplitudes. As Fig. 5 shows, rare particles are ejected to large amplitudes despite a pronounced dependence of the orbital period on the amplitude. The main point, the generality of which is highlighted by the addition of this "gravitational" example, is that colored noise combined with parametric resonance will drive a statistically small number of particles to much larger amplitudes than parametric resonance can do on its own. The formation of distended halos is thus a general by-product of collective relaxation of nonequilibrium Coulomb systems.

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[1] J. Gao, Nucl. Instrum. Methods Phys. Res., Sect. A **484**, 27 (2002).
[2] R. Jameson, Fusion Eng. Des. **32-33**, 149 (1996).
[3] Spallation Neutron Source Report No. 100000000-PL0001-R09 (unpublished).
[4] Y.-J. Chen *et al.*, in *Proceedings of the 1991 Particle Accelerator Conference*, edited by L. Lazema and J. Chew (IEEE, Piscataway, NJ, 1991), p. 3100.
[5] J. S. O'Connell, T. P. Wangler, R. S. Mills, and K. R. Crandell, in *Proceedings of the 1993 Particle Accelerator Conference*, edited by S. T. Corneliussen (IEEE, Piscataway, NJ, 1993), p. 3657.
[6] R. L. Gluckstern, Phys. Rev. Lett. **73**, 1247 (1994).
[7] H. Okamoto and M. Ikegami, Phys. Rev. E **55**, 4694 (1997).
[8] R. L. Gluckstern, A. V. Fedotov, S. Kurennoy, and R. Ryne, Phys. Rev. E **58**, 4977 (1998).
[9] M. Ikegami, Phys. Rev. E **59**, 2330 (1999).
[10] J. Qiang and R. Ryne, Phys. Rev. ST Accel. Beams **3**, 064201 (2000).
[11] T.-S. F. Wang, Phys. Rev. E **61**, 855 (2000).
[12] D. Jeon *et al.*, Phys. Rev. ST Accel. Beams **5**, 094201 (2002).
[13] C. K. Allen *et al.*, Phys. Rev. Lett. **89**, 214802 (2002).
[14] C. L. Bohn and I. V. Sideris, Phys. Rev. ST Accel. Beams **6**, 034203 (2003); $\Phi(r)$ is the spherically symmetric "case 5" potential of this reference. Given how Eq. (2) is normalized, a representative size of the matched beam is $R = 10$; thus, we rescaled $r(t)$ for Model II by 0.1 to facilitate comparing results with those of Model I.
[15] N. Brown and M. Reiser, Phys. Plasmas **2**, 965 (1995).
[16] I. V. Pogorelov and H. E. Kandrup, Phys. Rev. E **60**, 1567 (1999).
[17] H. E. Kandrup, I. V. Sideris, and C. L. Bohn, Phys. Rev. ST Accel. Beams (to be published).
[18] W. H. Press *et al.*, *Numerical Recipes* (Cambridge University Press, Cambridge, 1992); even with identical parameters, no two orbits are identical because the equation of motion includes randomness from the noise.
[19] J. Qiang *et al.*, Nucl. Instrum. Methods Phys. Res., Sect. A **457**, 1 (2001).
[20] J. Binney and S. Tremaine, *Galactic Dynamics* (Princeton University Press, Princeton, 1987), pp. 223–225.